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Radford, Drinfeld and Cardy boundary states in the (1, p) logarithmic conformal field models

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Abstract
We introduce $(p-1)$ pseudocharacters in the space of $(1, p)$ model vacuum torus amplitudes to complete the distinguished basis in the $2p$-dimensional fusion algebra to a basis in the whole $(3p-1)$-dimensional space of torus amplitudes, and the structure constants in this basis are (not necessarily non-negative) integer numbers. We obtain a generalized Verlinde formula that gives these structure constants. In the context of theories with boundaries, we identify the space of vacuum torus amplitudes with the space of Ishibashi states. Then, we propose $(3p-1)$ boundary states satisfying the Cardy condition.

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1. Introduction

1.0. Logarithmic conformal field theories [1–9] attract attention because of their significance for applications (like the sand-pile model [10, 11] and percolation [12–14]) as well as for general questions in the theory itself [15, 16]. Between logarithmic conformal field theories the $(1, p)$ models [17, 18] are studied in most detail [1, 4, 19]. The $(1, p)$ models are characterized by the central charge

$$c = 13 - 6(p + p^{-1}), \tag{1.1}$$

and the spectrum of conformal dimensions of primary fields is given by

$$\Delta_r = \frac{r^2 - 1}{4p} + \frac{1 - r}{2}, \qquad 1 \leqslant r \leqslant p.$$

The chiral symmetry algebra of the $(1, p)$ models is the triplet W algebra [20], which we denote in what follows as $\mathcal{W}_p$.

Logarithmic conformal field theories with boundaries are of great importance in considering their applications in lattice models, which usually involve boundaries [2, 21, 22].

Various attempts to understand boundary logarithmic theories have been made in the past [23–28] and recently [29, 30].

In this paper, we propose using the Kazhdan–Lusztig correspondence stated for some logarithmic models [5, 32, 33] in studying the boundary theories. The Kazhdan–Lusztig correspondence for the $(1, p)$ models is an equivalence [34] between the representation categories of the triplet algebra $\mathcal{W}_p$ and the *restricted* quantum group $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, with $\mathfrak{q} = e^{\frac{i\pi}{p}}$. These categories are equivalent to the tensor categories (see [31, 34, 36]). The representation category of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ is not braided [35], but is very close to braided (we discuss this subtlety in conclusion). This equivalence leads to the following identifications:

(1) the center $\mathsf{Z}_{\mathsf{cft}}$ of the $\mathcal{W}_p$ representation category and the center Z of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ are isomorphic as associative commutative algebras;
(2) the modular group actions on $\mathsf{Z}_{\mathsf{cft}}$ and Z are equivalent.

In what follows, we use several equivalent definitions of $\mathsf{Z}_{\mathsf{cft}}$. The most general definition of $\mathsf{Z}_{\mathsf{cft}}$ is given in categorical terms as endomorphisms of the identity functor. More practical definitions of $\mathsf{Z}_{\mathsf{cft}}$ are given for rational models in [37]. The space $\mathsf{Z}_{\mathsf{cft}}$ is spanned by central elements (or endomorphisms) of the bimodule structure on the full space of bulk states; we mean the bimodule structure with respect to the chiral and antichiral actions. As in rational cases [37], in the context of the boundary theories, these endomorphisms can be identified with the boundary states. Therefore, keeping in mind the identifications (1) and (2), the center Z of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ and some additional structures (multiplication, modular group action) on it allow describing the space of the boundary states for the $(1, p)$ models. The space $\mathsf{Z}_{\mathsf{cft}}$ can also be identified with the space of torus amplitudes as in [37, formula (2.6)]. The difference between the semisimple and logarithmic/nonsemisimple cases is that the boundary states corresponding to the different (irreducible and indecomposable) modules of a chiral algebra are not necessarily orthogonal (see (5.1)–(5.7)) and therefore $\mathsf{Z}_{\mathsf{cft}}$ is spanned not only by characters of irreducible representations but also by some additional functions.

1.1. Our main results.

The space $\mathsf{Z}_{\mathsf{cft}}$ is $(3p - 1)$ dimensional and has a distinguished basis of the vacuum torus amplitudes, i.e. of the characters

$$\chi_s^{\pm}(\tau) = \mathrm{Tr}_{\mathcal{X}^{\pm}(s)} q^{L_0 - \frac{c}{24}}, \qquad 1 \leqslant s \leqslant p, \tag{1.2}$$

and the pseudocharacters

$$\chi_s(\tau) = \mathrm{Tr}_{\mathcal{P}^+(s) \oplus \mathcal{P}^-(p-s)} \big(q^{L_0 - \frac{c}{24}} \mathbb{1}\big), \qquad 1 \leqslant s \leqslant p - 1,$$

where $\mathbb{1}$ is the logarithmic partner of the identity operator $\mathbf{1}$, and $\mathcal{P}^{\pm}(s)$ are the projective covers of the irreducible $\mathcal{W}_p$ representations $\mathcal{X}^{\pm}(s)$; here, we also set $q = e^{2i\pi\tau}$, where τ is the modular parameter. The space $\mathsf{Z}_{\mathsf{cft}}$ can be endowed with a commutative associative algebra structure, which we introduce below in section 2.5. The structure constants in $\mathsf{Z}_{\mathsf{cft}}$ on the basis of the characters and pseudocharacters are the *integer* numbers.

1.1.1. Proposition. *For* $1 \leqslant r, s \leqslant p$, *and* $\alpha, \beta = \pm$,

$$\chi_r^{\alpha} \chi_s^{\beta} = \sum_{\substack{t=|r-s|+1\\ \text{step}=2}}^{r+s-1} \widetilde{\chi}_t^{\alpha\beta}, \qquad \widetilde{\chi}_t^{\alpha} = \begin{cases} \chi_t^{\alpha}, & 1 \leqslant t \leqslant p, \\ \chi_{2p-t}^{\alpha} + 2\chi_{t-p}^{-\alpha}, & p+1 \leqslant t \leqslant 2p-1, \end{cases} \tag{1.3}$$

and, for $1 \leqslant r, s \leqslant p-1$,

$$\chi_r^+\chi_s = \sum_{\substack{t=|r-s|+1\\ \text{step}=2}}^{\substack{\min(r+s-1,\\ 2p-r-s-1)}} \chi_t, \qquad \chi_{p-r}^-\chi_s = -\sum_{\substack{t=|r-s|+1\\ \text{step}=2}}^{\substack{\min(r+s-1,\\ 2p-r-s-1)}} \chi_t, \qquad \chi_p^\pm\chi_s = 0, \quad \chi_r\chi_s = 0,$$

where we set $\chi_s^\pm = \chi_s^\pm(\tau)$ *and* $\chi_s = \chi_s(\tau)$ *for simplicity.*

Multiplication (1.3) was first derived in [38] as fusion of irreducible $\mathcal{W}_p$ representations and was subsequently shown in [32] to be the Grothendieck ring of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$. The elements $\chi_s^\pm$ span the fusion algebra $\mathsf{G} \subset \mathsf{Z}_{\mathsf{cft}}$; the pseudocharacters χ_s span a $(p-1)$ dimensional ideal in $\mathsf{Z}_{\mathsf{cft}}$.

The space $\mathsf{Z}_{\mathsf{cft}}$ admits the modular group action generated by

$$S : \tau \mapsto -1/\tau, \qquad T : \tau \mapsto \tau + 1.$$

The structure constants in proposition 1.1.1 are reproduced from the S-matrix action and we thus get a generalized Verlinde formula for the $(1, p)$ models.

1.1.2. Proposition. *The structure constants in* $\mathsf{Z}_{\mathsf{cft}}$ *with respect to the basis of the characters and pseudocharacters are given by*

$$N_{[r;\alpha][s;\beta]}^{[k;\gamma]} = \sum_{l=1}^{p+1}\sum_{\lambda=1}^{n_l} \frac{S_{[l;1]}^{\text{vac}} S_{[l;1]}^{[r;\alpha]} S_{[l;\lambda]}^{[s;\beta]} + S_{[l;1]}^{\text{vac}} S_{[l;\lambda]}^{[r;\alpha]} S_{[l;1]}^{[s;\beta]} - S_{[l;\lambda]}^{\text{vac}} S_{[l;1]}^{[r;\alpha]} S_{[l;1]}^{[s;\beta]}}{\left(S_{[l;1]}^{\text{vac}}\right)^2} S_{[k;\gamma]}^{[l;\lambda]}, \tag{1.4}$$

where $1 \leqslant r, s, k \leqslant p+1$, $1 \leqslant \alpha, \beta, \gamma \leqslant n_l \leqslant 3$, *and* $S_{[r;\alpha]}^{\text{vac}}$ *are the 'vacuum' row elements (the notations are introduced below in sections 2 and 4).*

We note that the structure constants $N_{[r;\alpha][s;\beta]}^{[k;\gamma]}$ are integers and are non-negative whenever α, β and γ are not equal to 1.

The Verlinde formula (1.4) can be considered as a generalization of the $(1, p)$ model Verlinde formulas derived in [38, 30] because (1.4) gives the structure constants in the whole space of torus amplitudes, in which the fusion algebra is a $2p$-dimensional subalgebra.

The generalized Verlinde formula (1.4) is based on the following findings. The multiplication in $\mathsf{Z}_{\mathsf{cft}}$ in the basis

$$\phi_s^\pm = S(\chi_s^\pm), \qquad \phi_s = S(\chi_s) \tag{1.5}$$

is block diagonal, and the structure constants are expressed in terms of the S-matrix vacuum row elements.

1.1.3. Proposition. *For* $1 \leqslant r, s \leqslant p-1$, *the only nonzero multiplications in* $\mathsf{Z}_{\mathsf{cft}}$ *with respect to the basis (1.5) are given by*

$$\phi_r\phi_r = \frac{1}{S_{[r;1]}^{\text{vac}}}\left(\phi_r - \frac{S_{[r;2]}^{\text{vac}}}{S_{[r;1]}^{\text{vac}}}\phi_r^+ - \frac{S_{[r;3]}^{\text{vac}}}{S_{[r;1]}^{\text{vac}}}\phi_{p-r}^-\right),$$

$$\phi_r\phi_r^+ = \frac{1}{S_{[r;1]}^{\text{vac}}}\phi_r^+, \qquad \phi_r\phi_{p-r}^- = \frac{1}{S_{[r;1]}^{\text{vac}}}\phi_{p-r}^-,$$

$$\phi_p^+\phi_p^+ = \frac{1}{S_{[p;1]}^{\text{vac}}}\phi_p^+, \qquad \phi_p^-\phi_p^- = \frac{1}{S_{[p+1;1]}^{\text{vac}}}\phi_p^-.$$

Here, $S_{[r;\alpha]}^{\text{vac}}$, *with* $\alpha = 1, 2, 3$, *are the 'vacuum' row elements (the notations are introduced below in section 2).*

1.1.4. Boundary states. We identify the space of boundary states with the space of vacuum torus amplitudes $\mathsf{Z}_{\mathrm{cft}}$. Then, we define $(3p-1)$ Cardy states as states that satisfy the (extended) Cardy condition:

$$\langle\!\langle[r;\alpha]\|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}\|[s;\beta]\rangle\!\rangle = \sum_{k=1}^{p+1}\sum_{\gamma=1}^{n_k} N_{[r;\alpha][s;\beta]}^{[k;\gamma]}\chi_{[k;\gamma]}(\widetilde{q}),$$

where $\widetilde{q} = \mathrm{e}^{-2\mathrm{i}\pi/\tau}$ and the integer structure constants $N_{[r;\alpha][s;\beta]}^{[k;\gamma]}$ are given in (1.4), with the notations introduced in sections 2 and 5. We note that $2p$ Cardy states corresponding to the Greek indices taking values 2 and 3 are usual Cardy states as in rational models, but $(p-1)$ states corresponding to the Greek indices equal to 1 are some new objects typical for LCFTs.

This paper is organized as follows. In section 2, we introduce the space $\mathsf{Z}_{\mathrm{cft}}$ of vacuum torus amplitudes for the $(1, p)$ models and recall the modular group action on $\mathsf{Z}_{\mathrm{cft}}$ in section 2.4, closely following to [38, 32]. In section 2.5, we introduce an associative algebra structure in the space of torus amplitudes. Section 3 is designed to compute the multiplications in $\mathsf{Z}_{\mathrm{cft}}$ using some quantum-group techniques. In section 3.1, we recall the quantum group $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ dual to the triplet algebra $\mathcal{W}_p$. In section 3.4, we calculate the multiplications in the center Z with respect to two distinguished bases related by the S-transformation from the modular group and we thus obtain the multiplications in $\mathsf{Z}_{\mathrm{cft}}$. In section 4, these results are then applied to obtain a generalized Verlinde formula for the $(1, p)$ models. With this information we then analyze the boundary states for the $(1, p)$ models in section 5. Conclusions are given in section 6. The appendices contain auxiliary or bulky facts and proofs.

Notations

We use the standard abuse of notation for characters: we write $\chi(\tau)$ for $\chi(\mathrm{e}^{2\mathrm{i}\pi\tau})$ and set in what follows

$$q = \mathrm{e}^{2\mathrm{i}\pi\tau}, \qquad \widetilde{q} = \mathrm{e}^{-2\mathrm{i}\pi/\tau}.$$

We set in the paper

$$\mathfrak{q} = \mathrm{e}^{\frac{\mathrm{i}\pi}{p}},$$

for an integer $p \geqslant 2$, and use the standard notation

$$[n] = \frac{\mathfrak{q}^n - \mathfrak{q}^{-n}}{\mathfrak{q} - \mathfrak{q}^{-1}}, \qquad n \in \mathbb{Z}, \quad [n]! = [1][2]\cdots[n], \quad n \in \mathbb{N}, \quad [0]! = 1$$

(without indicating the 'base' $\mathfrak{q}$ explicitly) and set

$$\begin{bmatrix} m \\ n \end{bmatrix} = \begin{cases} 0, & n < 0 \quad \text{or} \quad m-n < 0, \\ \dfrac{[m]!}{[n]![m-n]!} & \text{otherwise.} \end{cases}$$

For the Hopf algebras in general and for $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ specifically, we write Δ, ϵ and S for the comultiplication, counit and antipode, respectively.

We write x', x'' (Sweedler's notation) in $\Delta(x) = \sum_{(x)} x' \otimes x''$.

For a linear function β, we use the notation $\beta(?)$, where *?* indicates the position of its argument in more complicated constructions.

2. The space $\mathsf{Z}_{\mathsf{cft}}$ of $(1, p)$ model torus amplitudes

We briefly recall the definition of the $(1, p)$ logarithmic models and their chiral symmetry algebra in section 2.1. We introduce the space of vacuum torus amplitudes associated with the $(1, p)$ logarithmic models in section 2.3 and recall their modular properties in section 2.4. In section 2.5, we introduce multiplication in the space of torus amplitudes.

2.1. Logarithmic $(1, p)$ *models*

Here, we closely follow [38]. Logarithmic $(1, p)$ models are defined as kernels of certain screening operators, which commute with the Virasoro algebra. The actual symmetry of the theory is the maximal local algebra in this kernel. In a $(1, p)$ model, which is the kernel of the 'short' screening operator, see [38], this is the triplet W-algebra $\mathcal{W}_p$ studied in [20, 1]. The chiral algebra $\mathcal{W}_p$ has $2p$ irreducible representations $\mathcal{X}^{\pm}(s)$ and $2p$ projective covers $\mathcal{P}^{\pm}(s)$ [38] of the irreducibles ($1 \leqslant s \leqslant p$). The quantum group counterparts of these modules are defined in appendix B.

2.2. The logarithmic partner of **1**

In what follows, we need to define logarithmic partner $\mathbf{1\!l}$ of the identity operator $\mathbf{1}$. In conformal field theory, an operator–state correspondence is supposed. It means that a field corresponds to any state and any state can be obtained by the action of the zero mode of a field on the vacuum $|0\rangle$. In particular, $\mathbf{1}$ corresponds to the vacuum itself.

The state $|0\rangle$ is the highest weight vector of the irreducible module $\mathcal{X}^+(1)$, which is called the vacuum module. In the logarithmic $(1, p)$ models, $\mathcal{X}^+(1)$ is a submodule in the reducible but indecomposable logarithmic module $\mathcal{P}^+(1)$. The module $\mathcal{P}^+(1)$ is cyclic and can be generated from the vector $|\Omega\rangle$, which can be chosen in such a way that $|\Omega\rangle \to |0\rangle$ under the natural surjection $\mathcal{P}^+(1) \to \mathcal{X}^+(1)$.

Due to the operator–state correspondence there is a field $\mathbf{1\!l}(z)$ that corresponds to the state $|\Omega\rangle$:

$$|\Omega\rangle = \mathbf{1\!l}(z)\mathbf{1}|_{z\to 0}\,. \tag{2.1}$$

We call the field $\mathbf{1\!l}(z)$ logarithmic partner of $\mathbf{1}$. In what follows, we abuse notation and use the symbol $\mathbf{1\!l}$ for the field and its zero mode.

2.3. The space of $(1, p)$ *model torus amplitudes*

The space $\mathsf{Z}_{\mathsf{cft}}$ of the vacuum torus amplitudes is $(3p - 1)$ dimensional[1] and is spanned by $2p$ characters of irreducible representations $\mathcal{X}^{\pm}(s)$ of $\mathcal{W}_p$,

$$\chi_s^+(\tau) = \mathrm{Tr}_{\mathcal{X}^+(s)} \mathrm{e}^{2\mathrm{i}\pi\tau(L_0 - \frac{c}{24})}, \qquad \chi_s^-(\tau) = \mathrm{Tr}_{\mathcal{X}^-(s)} \mathrm{e}^{2\mathrm{i}\pi\tau(L_0 - \frac{c}{24})}, \qquad 1 \leqslant s \leqslant p, \tag{2.2}$$

given in (A.1) in terms of theta functions, and $(p - 1)$ pseudocharacters assigned to the projective modules $\mathcal{P}^+(s) \oplus \mathcal{P}^-(p - s)$,

$$\begin{aligned} \chi_s(\tau) &= a_0\tau\left(\frac{p-s}{p}\chi_s^+(\tau) - \frac{s}{p}\chi_{p-s}^-(\tau)\right) \\ &= \mathrm{Tr}_{\mathcal{P}^+(s)\oplus\mathcal{P}^-(p-s)}\big(\mathrm{e}^{2\mathrm{i}\pi\tau(L_0-\frac{c}{24})}\mathbf{1\!l}\big), \qquad 1 \leqslant s \leqslant p - 1, \end{aligned} \tag{2.3}$$

where $\mathbf{1\!l}$ is the logarithmic partner of the vacuum field (identity operator $\mathbf{1}$) and a_0 is an arbitrary nonzero constant. The pseudocharacters $\chi_s(\tau)$ are given in (A.2).

[1] See [39] for the case $p = 2$.

2.3.1. Remark. We note that the pseudocharacters (2.3) have no canonical normalization and are therefore defined up to a common nonzero constant a_0. However, the constant a_0 does not appear in final results.

We define a vector of the characters and the pseudocharacters:

$$\underbrace{(\chi_1(\tau), \chi_1^+(\tau), \chi_{p-1}^-(\tau), \dots, \chi_{p-1}(\tau), \chi_{p-1}^+(\tau), \chi_1^-(\tau),}_{3\times(p-1)} \chi_p^+(\tau), \chi_p^-(\tau)), \tag{2.4}$$

where we group the characters and pseudocharacters into $(p-1)$ triplets $\{\chi_s(\tau), \chi_s^+(\tau), \chi_{p-s}^-(\tau)\}$, for $1 \leqslant s \leqslant p-1$, and into two singlets $\chi_p^+(\tau)$ and $\chi_p^-(\tau)$.

In what follows, we use 2 -index parameterization of the elements in (2.4),

$$\begin{aligned} &(\chi_{[s;1]}(\tau), \chi_{[s;2]}(\tau), \chi_{[s;3]}(\tau)) = \big(\chi_s(\tau), \chi_s^+(\tau), \chi_{p-s}^-(\tau)\big), \qquad 1 \leqslant s \leqslant p-1,\\ &\chi_{[p;1]}(\tau) = \chi_p^+(\tau), \qquad \chi_{[p+1;1]}(\tau) = \chi_p^-(\tau). \end{aligned} \tag{2.5}$$

2.4. The modular group action on $\mathsf{Z}_{\mathrm{cft}}$

The $SL(2,\mathbb{Z})$ action in terms of the vector $(\chi_{[s;\alpha]}(\tau))$ in (2.5) can be written as follows:

$$\chi_{[s;\alpha]}(-1/\tau) = \sum_{j=1}^{p+1}\sum_{\beta=1}^{n_j} S_{[s;\alpha][j;\beta]}\chi_{[j;\beta]}(\tau), \qquad 1 \leqslant s \leqslant p-1,$$

where $n_j = 3$, for $1 \leqslant j \leqslant p-1$, and $n_j = 1$, for $j = p, p+1$, and the matrix $S_{[s;\alpha][j;\beta]}$ has the following block structure [32] (see also [38, 40]):

$$\begin{matrix} 3\times 3 & \cdots & 3\times 3 & 3\times 2\\ \vdots & \ddots & \vdots & \vdots\\ 3\times 3 & \cdots & 3\times 3 & 3\times 2\\ 2\times 3 & \cdots & 2\times 3 & 2\times 2 \end{matrix}, \tag{2.6}$$

where the 3×3 blocks, labeled by (s, j) with $s, j = 1, \dots, p-1$, are given by

$$\begin{pmatrix} S_{[s;1][j;1]} & S_{[s;1][j;2]} & S_{[s;1][j;3]}\\ S_{[s;2][j;1]} & S_{[s;2][j;2]} & S_{[s;2][j;3]}\\ S_{[s;3][j;1]} & S_{[s;3][j;2]} & S_{[s;3][j;3]} \end{pmatrix} = \frac{(-1)^{p+s+j}}{\sqrt{2p}} \begin{pmatrix} 0 & a_0\dfrac{p-j}{p}(\mathrm{q}^{sj}-\mathrm{q}^{-sj}) & -a_0\dfrac{j}{p}(\mathrm{q}^{sj}-\mathrm{q}^{-sj})\\ -\dfrac{1}{a_0}(\mathrm{q}^{sj}-\mathrm{q}^{-sj}) & \dfrac{s}{p}(\mathrm{q}^{sj}+\mathrm{q}^{-sj}) & \dfrac{s}{p}(\mathrm{q}^{sj}+\mathrm{q}^{-sj})\\ \dfrac{1}{a_0}(\mathrm{q}^{sj}-\mathrm{q}^{-sj}) & \dfrac{p-s}{p}(\mathrm{q}^{sj}+\mathrm{q}^{-sj}) & \dfrac{p-s}{p}(\mathrm{q}^{sj}+\mathrm{q}^{-sj}) \end{pmatrix}, \tag{2.7}$$

with $\mathrm{q} = \mathrm{e}^{\frac{\mathrm{i}\pi}{p}}$, the 3×2 blocks, labeled by (s, p) with $s = 1, \dots, p-1$, are

$$\begin{pmatrix} S_{[s;1][p;1]} & S_{[s;1][p+1;1]}\\ S_{[s;2][p;1]} & S_{[s;2][p+1;1]}\\ S_{[s;3][p;1]} & S_{[s;3][p+1;1]} \end{pmatrix} = \frac{1}{p\sqrt{2p}} \begin{pmatrix} 0 & 0\\ s & (-1)^{p-s}s\\ p-s & (-1)^{p-s}(p-s) \end{pmatrix}, \tag{2.8}$$

the 2×3 blocks, labeled by (p, j) with $j = 1, \dots, p-1$, are

$$\begin{pmatrix} S_{[p;1][j;1]} & S_{[p;1][j;2]} & S_{[p;1][j;3]}\\ S_{[p+1;1][j;1]} & S_{[p+1;1][j;2]} & S_{[p+1;1][j;3]} \end{pmatrix} = \frac{2}{\sqrt{2p}} \begin{pmatrix} 0 & 1 & 1\\ 0 & (-1)^{p-j} & (-1)^{p-j} \end{pmatrix}, \tag{2.9}$$

and the 2×2 block is given by

$$\begin{pmatrix} S_{[p;1][p;1]} & S_{[p;1][p+1;1]} \\ S_{[p+1;1][p;1]} & S_{[p+1;1][p+1;1]} \end{pmatrix} = \frac{1}{\sqrt{2p}} \begin{pmatrix} 1 & 1 \\ 1 & (-1)^p \end{pmatrix}. \tag{2.10}$$

The S-matrix has a distinguished row

$$\left(S^{\rm vac}_{[j;\beta]}\right) = \left(S_{[1;2][j;\beta]}\right), \tag{2.11}$$

which corresponds to the S-transformation of the vacuum representation character $\chi_1^+(\tau)$:

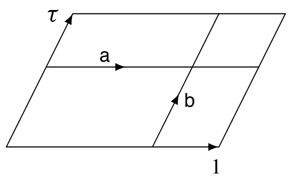
$$\chi_1^+(-1/\tau) = \sum_{j=1}^{p+1} \sum_{\beta=1}^{n_j} S^{\rm vac}_{[j;\beta]} \chi_{[j;\beta]}(\tau).$$

The T-transformation on the space of torus amplitudes is given in [38] and we do not reproduce the T-action here. In what follows, we need the properties of $\mathsf{Z}_{\mathsf{cft}}$ with respect to the S-transformation only.

2.5. Multiplication in the space of torus amplitudes

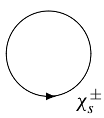
The space of vacuum torus amplitudes can be endowed with an associative commutative algebra structure in the way similar to the one in [41, 42] for semisimple (rational) cases. Here, we introduce such algebra structure on the space $\mathsf{Z}_{\mathsf{cft}}$ of torus amplitudes for the $(1, p)$ models. But the reader should note that we give only heuristic description.

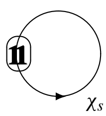
Let a and b denote two basic Dehn twists in a torus depicted in figure (2.12).



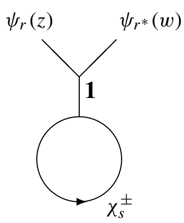
(2.12)

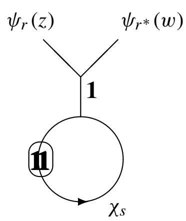
where τ is in the upper-half plane. The characters $\chi_s^\pm(\tau)$ in (1.2) and the pseudocharacters $\chi_s(\tau)$ in (2.3) are the conformal blocks of zero-point correlations on the torus. We depict these conformal blocks as

$\chi_s^\pm(\tau) =$ 

$\chi_s(\tau) =$ 

with the cycles corresponded to the b cycle in the torus. Next, we introduce the conformal blocks

$\mathcal{F}^{\pm,s}_{r,r*}(z-w) =$ 

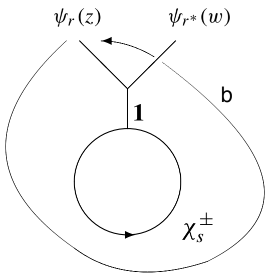
$\mathcal{F}^{s}_{r,r*}(z-w) =$ 

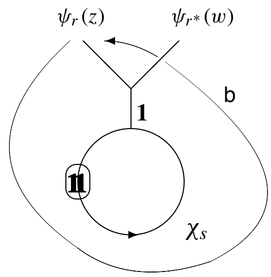
for two-point correlators $\langle\psi_r(z)\psi_{r^*}(w)\rangle$ with the self-conjugate primary fields $\psi_r(z)$ $(r = r^*)$ with respect to $\mathcal{W}_p$. The characters $\chi_s^{\pm}(\tau)$ and the pseudocharacters $\chi_s(\tau)$ can be obtained from these conformal blocks in the limit of coinciding points,

$$\chi_s^{\pm}(\tau) = \lim_{z\to w}(z-w)^{\Delta_r}\mathcal{F}^{\pm,s}_{r,r*}(z-w), \qquad \chi_s(\tau) = \lim_{z\to w}(z-w)^{\Delta_r}\mathcal{F}^{s}_{r,r*}(z-w),$$

where Δ_r is the conformal dimension of $\psi_r(z)$.

We define multiplication '$\star$' of a primary field $\psi_r(z)$ with $\chi_s^{\pm}(\tau)$ and $\chi_s(\tau)$ as a result of computing the monodromy, along the b cycle, of the conformal blocks $\mathcal{F}^{\pm,s}_{r,r*}(z-w)$ and $\mathcal{F}^{s}_{r,r*}(z-w)$ with respect to the z coordinate:

$\psi_r(z)\star\chi_s^{\pm}(\tau) =$ 

$\psi_r(z)\star\chi_s(\tau) =$ 

and taking the limit of coinciding points at the end of the computation. Then, the multiplication in the space Z_{cft} of the vacuum torus amplitudes is defined as follows:

$$\chi_r^{\pm}\chi_s^{\alpha} = \psi_r^{\pm}(z)\star\chi_s^{\alpha}(\tau), \qquad \chi_r^{\pm}\chi_s = \psi_r^{\pm}(z)\star\chi_s(\tau), \qquad \chi_r\chi_s = \psi_r(z)\star\chi_s(\tau),$$

where $\alpha\in\{+,-\}$, $\psi_r^{\pm}(z)$ are the primary fields of $\mathcal{X}^{\pm}(r)$, and $\psi_r(z)$ are the appropriate logarithmic partners.

The Kazhdan–Lusztig correspondence stated in [32] for the $(1, p)$ logarithmic models have led in [34] to an equivalence between representation categories of the chiral algebra $\mathcal{W}_p$ and of $\overline{\mathcal{U}}_{\mathrm{q}}s\ell(2)$. Such remarkable correspondence suggests an isomorphism between the space Z_{cft} of torus amplitudes in the $(1, p)$ models and the center Z of $\overline{\mathcal{U}}_{\mathrm{q}}s\ell(2)$ as unital commutative associative algebras. We compute the introduced multiplication in Z_{cft} below in section 3.4 using a quantum-group approach.

3. The quantum group center Z

To describe Z_{cft} as an associative commutative algebra, we first recall some facts about the center Z of the restricted quantum group $\overline{\mathcal{U}}_{\mathrm{q}}s\ell(2)$. As was shown in [32], the center Z is $(3p-1)$ dimensional and admits an $SL(2,\mathbb{Z})$ action in the following way. The space Z contains two special bases. The first one consists of the images of characters and pseudocharacters (those associated with some indecomposable representations) under the Drinfeld mapping $\boldsymbol{\chi}$ [43] and the second one under the Radford mapping $\boldsymbol{\phi}$ [44]. We call these bases *the Drinfeld and Radford bases*, respectively. Then, the $\mathcal{S}$-transformation from the modular group maps each vector from the Drinfeld basis to a vector from the Radford basis and vice versa [46, 47, 51]. Schematically, the modular group action on the center is given by the diagrams

$$\mathcal{S}:\ \begin{tikzcd} & \mathsf{Ch} \arrow[dl, "\boldsymbol{\chi}"'] \arrow[dr, "\boldsymbol{\phi}"] & \\ \mathsf{Z} \arrow[rr, leftrightarrow, bend right, "\mathcal{S}"'] & & \mathsf{Z} \end{tikzcd} \qquad \mathcal{T}:\ \mathsf{Z}\xrightarrow{\boldsymbol{v}}\mathsf{Z},$$

where $\mathcal{S} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $\mathcal{T} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and Ch is the space of characters and pseudocharacters, and $\boldsymbol{v}$ is a ribbon element.

As we show below in section 3.4, the structure constants in Z with respect to the Drinfeld basis are integer numbers, while in the Radford basis the multiplication is block diagonal, and the structure constants are expressed in terms of the S-matrix vacuum row elements. We then use this information and the isomorphism [32] between Z and $\mathsf{Z}_{\mathsf{cft}}$ as $SL(2,\mathbb{Z})$ representations in section 4 to obtain the generalized Verlinde formula (1.4) for the $(1, p)$ models.

3.1. The definition of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$

The quantum group dual to the $(1, p)$ logarithmic model with the chiral algebra $\mathcal{W}_p$ is the 'restricted' quantum $s\ell(2)$ denoted as $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ [32] with

$$\mathfrak{q} = e^{\frac{i\pi}{p}}.$$

The three generators E, F and K satisfy the standard relations for the quantum $s\ell(2)$,

$$KEK^{-1} = \mathfrak{q}^2 E, \qquad KFK^{-1} = \mathfrak{q}^{-2} F, \qquad [E, F] = \frac{K - K^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$

with some additional constraints,

$$E^p = F^p = 0, \qquad K^{2p} = \mathbf{1},$$

and the Hopf-algebra structure is given by

$$\Delta(E) = \mathbf{1} \otimes E + E \otimes K, \qquad \Delta(F) = K^{-1} \otimes F + F \otimes \mathbf{1}, \qquad \Delta(K) = K \otimes K,$$
$$\epsilon(E) = \epsilon(F) = 0, \qquad \epsilon(K) = 1,$$
$$S(E) = -EK^{-1}, \quad S(F) = -KF, \qquad S(K) = K^{-1}.$$

The elements of the PBW basis of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ are enumerated as $E^i K^j F^\ell$ with $0 \leqslant i \leqslant p-1$, $0 \leqslant j \leqslant 2p-1$, $0 \leqslant \ell \leqslant p-1$, and its dimension is therefore $2p^3$.

3.2. The quantum group center

Here, we describe the center Z of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ in terms of the Drinfeld basis and the Radford basis. To construct these bases, we first use the irreducible and projective modules over $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ to produce the space $\mathsf{Ch} = \mathsf{Ch}(\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2))$ of q-characters, which is dual to the center. The irreducible and projective modules are defined in appendix B. Then, we obtain two distinguished bases in the center related to the $\mathcal{S}$-transformation. In order to be self-contained we begin by reviewing some standard facts about the space Ch of q-characters, and the Radford and Drinfeld maps, following [32, 33].

3.2.1. The space of q-characters for $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$. For a Hopf algebra A, the space $\mathsf{Ch} = \mathsf{Ch}(A)$ of q-characters is defined as

$$\begin{aligned} \mathsf{Ch}(A) &= \{\beta \in A^* | \mathrm{Ad}^*_x(\beta) = \epsilon(x)\beta \ \forall x \in A\} \\ &= \{\beta \in A^* | \beta(xy) = \beta\big(S^2(y)x\big) \ \forall x, y \in A\}, \end{aligned} \tag{3.1}$$

where the co-adjoint action $\mathrm{Ad}^*_a : A^* \to A^*$ is $\mathrm{Ad}^*_a(\beta) = \beta\big(\sum_{(a)} S(a')?a''\big)$, $a \in A$, $\beta \in A^*$ and S is the antipode.

In what follows, we need the so-called *balancing element* $\boldsymbol{g} \in A$ that satisfies [43]

$$\Delta(\boldsymbol{g}) = \boldsymbol{g} \otimes \boldsymbol{g}, \qquad S^2(x) = \boldsymbol{g} x \boldsymbol{g}^{-1}, \tag{3.2}$$

for all $x \in A$. For $A = \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, $\boldsymbol{g} = K^{p+1}$ (see [32]).

3.2.2. Irreducible representation traces. The space of q-characters contains a homomorphic image of the Grothendieck ring under the q-trace: for any A-module X,

$$\mathrm{qCh}_{\mathsf{X}} \equiv \mathrm{Tr}_{\mathsf{X}}(\boldsymbol{g}^{-1}?) \in \mathsf{Ch}(A), \tag{3.3}$$

where $\boldsymbol{g}$ is the balancing element (3.2). For $A = \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, we thus have a $2p$-dimensional subspace in Ch spanned by q-traces over irreducible modules, i.e. by

$$\begin{aligned} &\gamma^{\pm}(s) : x \mapsto \mathrm{Tr}_{\mathsf{X}^{\pm}_s}(\boldsymbol{g}^{-1}x), \qquad 1 \leqslant s \leqslant p, \\ &\gamma^{\pm}(s) \in \mathsf{Ch}, \end{aligned} \tag{3.4}$$

with $\boldsymbol{g}^{-1} = K^{p-1}$.

3.2.3. Pseudotraces. The space of q-characters $\mathsf{Ch}(\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2))$ is not exhausted by q-traces over irreducible modules; it also contains 'pseudotraces' associated with the projective modules. To construct the pseudotraces (they can be considered quantum group counterparts of pseudotraces in [45] and of pseudocharacters in (2.3)), we closely follow the strategy proposed in [33]. We first consider the maps

$$\sigma_s : \mathsf{P}^+_s \oplus \mathsf{P}^-_{p-s} \to \mathsf{P}^+_s \oplus \mathsf{P}^-_{p-s}, \tag{3.5}$$

defined by its action on the corresponding basis vectors (see appendix B.1 and appendix B.2): σ_s acts by zero on all basis elements except

$$\sigma_s : \mathsf{b}^{(\pm,s)}_n \mapsto \alpha^{\pm}_s \mathsf{t}^{(\pm,s)}_n + \beta^{\pm}_s \mathsf{b}^{(\pm,s)}_n, \tag{3.6}$$

with the arbitrary coefficients $\beta^{\pm}_s$ and $\alpha^{\pm}_s \neq 0$. The map σ_s has the diagonal part: $\mathsf{b}^{(\pm,s)}_n \mapsto \beta^{\pm}_s \mathsf{b}^{(\pm,s)}_n$, and the nondiagonal part: $\mathsf{b}^{(\pm,s)}_n \mapsto \alpha^{\pm}_s \mathsf{t}^{(\pm,s)}_n$ corresponding to the action from the bottom (the socle) to the top of the projective module $\mathsf{P}^+_s \oplus \mathsf{P}^-_{p-s}$.

For any such σ_s, we now define a functional on $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ as

$$\gamma(s) : x \mapsto \mathrm{Tr}_{\mathsf{P}^+_s \oplus \mathsf{P}^-_{p-s}}(\boldsymbol{g}^{-1} x \sigma_s). \tag{3.7}$$

3.2.4. Proposition. *For* $1 \leqslant s \leqslant p-1$,

$$\gamma(s) \in \mathsf{Ch}$$

if and only if

$$\alpha^+_s = \alpha^-_s \tag{3.8}$$

The proof is similar to the one in [33], proposition 2.3.4.

3.2.5. Remark. We make two comments on the operators σ_s that appear in the definition of the pseudotraces $\gamma(s)$. First, we emphasize that σ_s is not an intertwiner of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ modules. We use these operators to reach nondiagonal elements of x in (3.7) like it is done in [45], where the idea of the pseudotraces is based on the fact that the functional $f(x) = \mathrm{Tr}\big(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x\big)$ gives zero value on diagonal matrices x and gives 1 on $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Second, a choice of the coefficients $\alpha^{\pm}_s$ and $\beta^{\pm}_s$ excepting (3.8) is a question of convenience. Nonzero $\beta^{\pm}_s$ leads to the addition of $\gamma^{\pm}(s)$ to $\gamma(s)$ and the choice of $\alpha^{\pm}_s$ is a normalization of $\gamma(s)$, which is not canonical because $\gamma(s)$ is nilpotent and in the following subsection we choose it from the identification of Z with $\mathsf{Z}_{\mathrm{cft}}$.

3.2.6. The γ basis. The space Ch of q-characters of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ is spanned by the elements

$$\gamma(s), \quad \gamma^{+}(s), \quad \gamma^{-}(p-s), \quad 1 \leqslant s \leqslant p-1, \quad \gamma^{+}(p), \quad \gamma^{-}(p), \tag{3.9}$$

where $\gamma^{\pm}(s)$ are defined in (3.4) and $\gamma(s)$ are defined in (3.7) with σ_s in the definition (3.6) fixed as

$$\sigma_s : \mathsf{b}_n^{(\pm,s)} \mapsto \alpha_s \mathsf{t}_n^{(\pm,s)}, \qquad \alpha_s = -a_0 \frac{[s]}{p(\mathfrak{q}-\mathfrak{q}^{-1})}, \tag{3.10}$$

where we fix the diagonal part of σ_s as zero, and a_0 is the normalization constant from (2.3). We thus have $(3p-1)$ linear independent q-characters. In what follows, we call these elements the *γ basis*.

3.2.7. Radford map of $\mathsf{Ch}(\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2))$ For a Hopf algebra A with the right integral $\boldsymbol{\mu}$ and the left–right cointegral $\boldsymbol{c}$ (see the definitions in [32] and references therein), *the Radford map* $\boldsymbol{\phi} : A^* \to A$ and its inverse $\boldsymbol{\phi}^{-1} : A \to A^*$ are given by

$$\boldsymbol{\phi}(\beta) = \sum_{(\boldsymbol{c})} \beta(\boldsymbol{c}')\boldsymbol{c}'', \qquad \boldsymbol{\phi}^{-1}(x) = \boldsymbol{\mu}(S(x)?). \tag{3.11}$$

We now calculate the Radford map $\boldsymbol{\phi} : \mathsf{Ch} \to \mathsf{Z}$ on the γ basis (3.9) in $\mathsf{Ch}(\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2))$ to obtain the Radford basis in the center of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$:

- The Radford map $\boldsymbol{\phi}$ on the irreducible representation traces $\gamma^{\pm}(s)$ is given by

$$\boldsymbol{\phi}(\gamma^{\pm}(s)) = \boldsymbol{\phi}^{\pm}(s) = \sum_{(\boldsymbol{c})} \mathrm{Tr}_{\mathsf{X}_s^{\pm}}(K^{p-1}\boldsymbol{c}')\boldsymbol{c}'',$$

where the left–right cointegral $\boldsymbol{c}$ is given by [32]

$$\boldsymbol{c} = \zeta F^{p-1} E^{p-1} \sum_{j=0}^{2p-1} K^j \tag{3.12}$$

with the normalization $\zeta = \sqrt{\frac{p}{2}} \frac{1}{([p-1]!)^2}$ and $\boldsymbol{\mu}(\boldsymbol{c}) = 1$. In [32], $\boldsymbol{\phi}^{\pm}(s)$ were calculated in the PBW basis,

$$\boldsymbol{\phi}^{\pm}(s) = \zeta \sum_{n=0}^{s-1} \sum_{i=0}^{n} \sum_{j=0}^{2p-1} (\pm 1)^{\mathrm{i}+\mathrm{j}} ([\mathrm{i}]!)^2 \mathfrak{q}^{j(s-1-2n)} \begin{bmatrix} s-n+i-1 \\ \mathrm{i} \end{bmatrix} \begin{bmatrix} n \\ \mathrm{i} \end{bmatrix} F^{p-1-i} E^{p-1-i} K^j, \tag{3.13}$$

- The Radford map $\boldsymbol{\phi}$ on the pseudotraces $\gamma(s)$ is given by

$$\boldsymbol{\phi}(\gamma(s)) = \boldsymbol{\phi}(s) = \sum_{(\boldsymbol{c})} \mathrm{Tr}_{\mathsf{P}_s^{+} \oplus \mathsf{P}_{p-s}^{-}}(K^{p-1}\boldsymbol{c}'\sigma_s)\boldsymbol{c}'',$$

where the map σ_s is defined in (3.5) and (3.10). In appendix C.3, we evaluate $\boldsymbol{\phi}(s)$ as

$$\begin{aligned} \boldsymbol{\phi}(s) = \alpha_s \zeta \sum_{m=0}^{p-2} \sum_{j=0}^{2p-1} \Bigg(& \sum_{n=0}^{s-1} q^{j(s-1-2n)} \mathsf{B}^{+}_{n,p-1-m}(s) \\ & + \sum_{k=0}^{p-s-1} q^{j(-s-1-2k)} \mathsf{B}^{-}_{k,p-1-m}(p-s) \Bigg) F^m E^m K^j, \qquad 1 \leqslant s \leqslant p-1, \end{aligned} \tag{3.14}$$

with α_s given in (3.10), and the coefficients $\mathsf{B}^{+}_{n,m}(s)$ and $\mathsf{B}^{-}_{k,m}(p-s)$ are given in (C.4), (C.5), and (C.6).

3.2.8. Drinfeld map of $\mathsf{Ch}(\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2))$. For a Hopf algebra A with the nondegenerate M-matrix ($M = R_{21}R_{12} \in A \otimes A$), which satisfies the relations

$$(\Delta \otimes \mathrm{id})(M) = R_{32}M_{13}R_{23}, \qquad M\Delta(x) = \Delta(x)M \quad \forall x \in A,$$

the Drinfeld map $\chi : A^* \to A$ is defined by

$$\chi(\beta) = (\beta \otimes \mathrm{id})M. \tag{3.15}$$

We emphasize that the Drinfeld map is defined for any Hopf algebra with a nondegenerate M-matrix (not necessarily quasi-triangular). In a Hopf algebra A with the nondegenerate M-matrix, the restriction of the Drinfeld map to the space Ch of q-characters gives an isomorphism $\mathsf{Ch}(A) \xrightarrow{\simeq} \mathsf{Z}(A)$ of associative algebras [43].

We now calculate the Drinfeld map $\chi : \mathsf{Ch} \to \mathsf{Z}$ on the γ basis (3.9) in $\mathsf{Ch}(\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2))$ to obtain the Drinfeldbasis in the center of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$:

- The Drinfeld map χ on the irreducible representation traces $\gamma^{\pm}(s)$ is given by

$$\chi(\gamma^{\pm}(s)) = \chi^{\pm}(s) = \big(\mathrm{Tr}_{\mathsf{X}_s^{\pm}} \otimes \mathrm{id}\big)((K^{p-1} \otimes \mathbf{1})M), \qquad 1 \leqslant s \leqslant p,$$

where the M-matrix is given by [32]

$$M = \frac{1}{2p}\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\sum_{i=0}^{2p-1}\sum_{j=0}^{2p-1}\frac{(\mathfrak{q}-\mathfrak{q}^{-1})^{m+n}}{[m]![n]!}\mathfrak{q}^{m(m-1)/2+n(n-1)/2}$$
$$\times \mathfrak{q}^{-m^2-mj+2nj-2ni-ij+mi}F^mE^nK^j \otimes E^mF^nK^i. \tag{3.16}$$

From ([32], proposition 4.3.1), we have

$$\chi^{\alpha}(s) = \alpha^{p+1}(-1)^{s+1}\sum_{n=0}^{s-1}\sum_{m=0}^{n}(\mathfrak{q}-\mathfrak{q}^{-1})^{2m}\mathfrak{q}^{-(m+1)(m+s-1-2n)}$$
$$\times \begin{bmatrix} s-n+m-1 \\ m \end{bmatrix}\begin{bmatrix} n \\ m \end{bmatrix} E^mF^mK^{s-1+\beta p-2n+m}, \tag{3.17}$$

where $1 \leqslant s \leqslant p$, $\alpha = \pm 1$, and we set $\beta = 0$ if $\alpha = +1$ and $\beta = 1$ if $\alpha = -1$.

- The Drinfeld map χ on the pseudotraces $\gamma(s)$ is given by

$$\chi(\gamma(s)) = \chi(s) = (\mathrm{Tr}_{\mathsf{P}_s^{+}\oplus\mathsf{P}_{p-s}^{-}} \otimes \mathrm{id})((K^{p-1} \otimes \mathbf{1})M(\sigma_s \otimes \mathrm{id})), \qquad 1 \leqslant s \leqslant p-1,$$

where the map σ_s is defined in (3.5) and (3.10). From (3.16), we obtain $\chi(s)$ by direct calculation,

$$\chi(s) = \alpha_s\sum_{m=1}^{p-1}(-1)^{s-1}\frac{(\mathfrak{q}-\mathfrak{q}^{-1})^{2m}}{([m]!)^2}\left(\sum_{n=0}^{s-1}q^{-m(m+s-2n)-(s-1-2n)}\mathsf{B}^{+}_{n,m}(s)K^{m+s-1-2n}\right.$$
$$\left. + \sum_{k=0}^{p-s-1}q^{-m(m-s-2k)+s+1+2k}\mathsf{B}^{-}_{k,m}(p-s)K^{m-s-1-2k}\right)E^mF^m, \tag{3.18}$$

with α_s given in (3.10) and the coefficients $\mathsf{B}^{+}_{n,m}(s)$ and $\mathsf{B}^{-}_{k,m}(p-s)$ are given in (C.4)–(C.6).

3.3. The modular group action on the center

The $SL(2,\mathbb{Z})$ action on the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ center Z is defined as [32, 33]

$$\begin{aligned} &\mathcal{S} : a \mapsto \boldsymbol{\phi}(\chi^{-1}(a)),\\ &\mathcal{T} : a \mapsto \mathrm{e}^{-2\mathrm{i}\pi\frac{c}{24}}\mathcal{S}(\boldsymbol{v}\mathcal{S}^{-1}(a)), \end{aligned} \tag{3.19}$$

which follows [46–48]. Here, the Drinfeld map χ and the Radford map $\boldsymbol{\phi}$ are defined in (3.15) and (3.11), respectively, the central charge c of $(1,p)$ models is given in (1.1), and the ribbon central element $\boldsymbol{v}$ is given in [32]. For the ribbon element we also have the remarkable expression $\boldsymbol{v} = \mathrm{e}^{2\mathrm{i}\pi L_0}$, where L_0 is the zero mode of the energy–momentum tensor for the $(1,p)$ models (see [32, 33]).

There are two bases in Z. The first one is the Radford basis

$$\{\boldsymbol{\phi}^{\pm}(s), \boldsymbol{\phi}(r) | 1 \leqslant s \leqslant p, 1 \leqslant r \leqslant p-1\} \tag{3.20}$$

evaluated in (3.13) and (3.14), and the second one is the Drinfeld basis

$$\{\chi^{\pm}(s), \chi(r) | 1 \leqslant s \leqslant p, 1 \leqslant r \leqslant p-1\} \tag{3.21}$$

evaluated in (3.17) and (3.18). These bases are related by the $\mathcal{S}$-transformation,

$$\mathcal{S}(\chi^{\pm}(r)) = \boldsymbol{\phi}^{\pm}(r), \qquad \mathcal{S}(\chi(r)) = \boldsymbol{\phi}(r). \tag{3.22}$$

3.4. Theorem. *The only nonzero multiplications in* Z *with respect to*

- *the Drinfeld basis (3.21) are given by*[2]*, for* $1 \leqslant r, s \leqslant p$*, and* $\alpha, \beta = \pm$*,*

$$\chi^{\alpha}(r)\chi^{\beta}(s) = \sum_{\substack{t=|r-s|+1\\ \text{step}=2}}^{r+s-1} \widetilde{\chi}^{\alpha\beta}(t), \tag{3.23}$$

where

$$\widetilde{\chi}^{\alpha}(t) = \begin{cases} \chi^{\alpha}(t), & 1 \leqslant t \leqslant p,\\ \chi^{\alpha}(2p-t) + 2\chi^{-\alpha}(t-p), & p+1 \leqslant t \leqslant 2p-1, \end{cases}$$

and, for $1 \leqslant r, s \leqslant p-1$*,*

$$\chi^{+}(r)\chi(s) = \sum_{\substack{i=|r-s|+1\\ \text{step}=2}}^{\min(r+s-1,2p-r-s-1)} \chi(i), \tag{3.24}$$

$$\chi^{-}(p-r)\chi(s) = -\sum_{\substack{i=|r-s|+1\\ \text{step}=2}}^{\min(r+s-1,2p-r-s-1)} \chi(i). \tag{3.25}$$

- *the Radford basis (3.20) are given by, for* $1 \leqslant r, s \leqslant p-1$*,*

$$\boldsymbol{\phi}(r)\boldsymbol{\phi}(r) = \frac{1}{S^{\text{vac}}_{[r;1]}}\left(\boldsymbol{\phi}(r) - \frac{S^{\text{vac}}_{[r;2]}}{S^{\text{vac}}_{[r;1]}}\boldsymbol{\phi}^{+}(r) - \frac{S^{\text{vac}}_{[r;3]}}{S^{\text{vac}}_{[r;1]}}\boldsymbol{\phi}^{-}(p-r)\right), \tag{3.26}$$

$$\boldsymbol{\phi}(r)\boldsymbol{\phi}^{+}(r) = \frac{1}{S^{\text{vac}}_{[r;1]}}\boldsymbol{\phi}^{+}(r), \qquad \boldsymbol{\phi}(r)\boldsymbol{\phi}^{-}(p-r) = \frac{1}{S^{\text{vac}}_{[r;1]}}\boldsymbol{\phi}^{-}(p-r), \tag{3.27}$$

[2] Multiplication (3.23) was calculated in [32] and was first obtained in [38] in the $(1,p)$ models Verlinde-formula context.

$$\boldsymbol{\phi}^{+}(p)\boldsymbol{\phi}^{+}(p) = \frac{1}{S^{\text{vac}}_{[p;1]}}\boldsymbol{\phi}^{+}(p), \qquad \boldsymbol{\phi}^{-}(p)\boldsymbol{\phi}^{-}(p) = \frac{1}{S^{\text{vac}}_{[p+1;1]}}\boldsymbol{\phi}^{-}(p). \tag{3.28}$$

Here, $\left(S^{\text{vac}}_{[r;\beta]}\right) = (S_{[1;2][r;\beta]})$, $1 \leqslant r \leqslant p+1$, $1 \leqslant \beta \leqslant n_r$, *is the 'vacuum' row, which corresponds to the* $\mathcal{S}$*-transformation of the unity* $\boldsymbol{\chi}^{+}(1) = \mathbf{1}$ *and* $S_{[1;2][r;\beta]}$ *are defined in (2.7) and (2.8).*

The proof of theorem 3.4 is given in appendix C.5.

It is useful to emphasize that all structure constants in (3.23) and (3.24) are non-negative integers and in (3.25) are non-positive integers.

Next, we recall that the space Z_{cft} of torus amplitudes and the center Z of the quantum group are identified as the modular group representations [32] and are isomorphic as associative commutative algebras. Therefore, identifying the elements in Z_{cft} and Z with the same modular-transformation properties

$$\begin{aligned} &\boldsymbol{\chi}^{\pm}(s) \mapsto \chi^{\pm}_s(\tau), && \boldsymbol{\chi}(s) \mapsto \chi_s(\tau),\\ &\boldsymbol{\phi}^{\pm}(s) \mapsto \chi^{\pm}_s(-1/\tau), && \boldsymbol{\phi}(s) \mapsto \chi_s(-1/\tau), \end{aligned}$$

we thus get two special bases in Z_{cft}. The first one is the basis of the characters and pseudocharacters given in (1.2) and (2.3), respectively, which corresponds to the Drinfeld basis in Z. The second one is the basis (1.5), which corresponds to the Radford basis in Z. We thus obtain proposition 1.1.1 using (3.23)–(3.25), and proposition 1.1.3 using (3.26)–(3.28).

3.4.1. Remark. The structure constants in (3.24) coincide with the structure constants in $\widehat{s\ell}(2)_k$ fusion algebra at the level $k=p-2$. We also note that the multiplications similar to the ones in (3.24) and (3.25) appear in [49] but for a different basis in the space of vacuum torus amplitudes.

4. Fusion rules and a generalized Verlinde formula

Here, we first give notes toward the generalization of the classical Verlinde formula [50] to nonsemisimple cases. Then, we use the results in 3.4 to obtain a generalized Verlinde formula for the $(1, p)$ models.

4.1. Toward generalization of the Verlinde formula

We assume that the S-transformation from the modular group acting on a finite-dimensional associative algebra, which we denote as Z_{cft}, satisfy

$$S^2|_{\mathsf{Z}_{\text{cft}}} = \text{id},$$

and block diagonalizes the structure constants (the 'fusion' coefficients) in Z_{cft} with respect to a distinguished basis $\{\chi_i | i \in I\}$, where I is some finite set, to the block-diagonal structure

$$n_1 \times n_1 \oplus \cdots \oplus n_r \times n_r \oplus \cdots \oplus n_P \times n_P, \tag{4.1}$$

where P is the number of the Jordan blocks, $1 \leqslant r \leqslant P$, and $n_r \geqslant 1$ is the rank of the rth Jordan block. We also assume that we have the unity $\mathbf{1}$ in Z_{cft}.

We note that these assumptions are satisfied in $(1, p)$ models.

We group the distinguished basis elements $\{\chi_i | i \in I\}$ in Z_{cft} into blocks (sets) with respect to the block-diagonal structure (4.1),

$$\left\{\chi_{[r;\alpha]} \equiv \chi_{n_1+\cdots+n_{r-1}+\alpha} \middle| 1 \leqslant r \leqslant P, 1 \leqslant \alpha \leqslant n_r\right\}.$$

We also arrange the structure constants, describing the multiplication between $\chi_{[r;\alpha]}$ and $\chi_{[s;\beta]}$ with the result in $[k;\gamma]$ block, into a rank-3 tensor $N_{[r;\alpha][s;\beta]}^{[k;\gamma]}$,

$$\chi_{[r;\alpha]}\chi_{[s;\beta]} = \sum_{k=1}^{P}\sum_{\gamma=1}^{n_k} N_{[r;\alpha][s;\beta]}^{[k;\gamma]}\chi_{[k;\gamma]}. \tag{4.2}$$

For $1 \leqslant r \leqslant P$ and $1 \leqslant \alpha \leqslant n_r$, we denote

$$\boldsymbol{\phi}_{[r;\alpha]} = S(\chi_{[r;\alpha]}) = \sum_{j=1}^{P}\sum_{\beta=1}^{n_j} S_{[r;\alpha][j;\beta]}\chi_{[j;\beta]},$$

and the structure constants in this basis by $\boldsymbol{\phi}_{[r;\lambda][s;\mu]}^{[k;\nu]}$,

$$\boldsymbol{\phi}_{[r;\lambda]}\boldsymbol{\phi}_{[s;\mu]} = \sum_{k=1}^{P}\sum_{\nu=1}^{n_k} \boldsymbol{\phi}_{[r;\lambda][s;\mu]}^{[k;\nu]}\boldsymbol{\phi}_{[k;\nu]}. \tag{4.3}$$

4.1.1. Theorem. *The structure constants $N_{[r;\alpha][s;\beta]}^{[k;\gamma]}$ in $\mathsf{Z}_{\mathsf{cft}}$ are reproduced from the S-matrix action,*

$$N_{[r;\alpha][s;\beta]}^{[k;\gamma]} = \sum_{l=1}^{P}\sum_{\lambda,\mu,\nu=1}^{n_l} S_{[r;\alpha][l;\lambda]}S_{[s;\beta][l;\mu]}\boldsymbol{\phi}_{[l;\lambda][l;\mu]}^{[l;\nu]}S_{[l;\nu][k;\gamma]}, \tag{4.4}$$

where the structure constants $\boldsymbol{\phi}_{[l;\lambda][l;\mu]}^{[l;\nu]}$ are solutions of the following equations, for $1 \leqslant l \leqslant P$ and $1 \leqslant \mu, \nu \leqslant n_l$,

$$\sum_{\lambda=1}^{n_l} S_{[l;\lambda]}^{\mathrm{vac}}\boldsymbol{\phi}_{[l;\lambda][l;\mu]}^{[l;\nu]} = \delta_{\mu,\nu}. \tag{4.5}$$

Here, $\left(S_{[l;\lambda]}^{\mathrm{vac}}\right) = (S_{[1;2][l;\lambda]})$, $1 \leqslant l \leqslant P$, $1 \leqslant \lambda \leqslant n_l$, is the 'vacuum' row, which corresponds to the S-transformation of the unity $\chi_2 = \mathbf{1}$.

Proof. Formula (4.4) is just a relation between the structure constants in different bases related by the S-transformation.

Let us consider the S-transformation of the unity. We have

$$S(\mathbf{1}) = \sum_{i=1}^{P}\sum_{\lambda=1}^{n_i} S_{[i;\lambda]}^{\mathrm{vac}}\chi_{[i;\lambda]}.$$

Next, we recall the assumption $S^2|_{\mathsf{Z}_{\mathsf{cft}}} = \mathrm{id}$. Therefore, we have

$$\mathbf{1} = \sum_{i=1}^{P}\sum_{\lambda=1}^{n_i} S_{[i;\lambda]}^{\mathrm{vac}}\boldsymbol{\phi}_{[i;\lambda]}.$$

Hence, using (4.3) and the assumption that the S-transformation block diagonalizes the structure constants in $\mathsf{Z}_{\mathsf{cft}}$ to the block-diagonal structure (4.1), we obtain the identities

$$\boldsymbol{\phi}_{[l;\mu]} = \sum_{i=1}^{P}\sum_{\lambda=1}^{n_i} S_{[i;\lambda]}^{\mathrm{vac}}\boldsymbol{\phi}_{[i;\lambda]}\boldsymbol{\phi}_{[l;\mu]} = \sum_{\lambda,\nu=1}^{n_l} S_{[l;\lambda]}^{\mathrm{vac}}\boldsymbol{\phi}_{[l;\lambda][l;\mu]}^{[l;\nu]}\boldsymbol{\phi}_{[l;\nu]},$$

for $1 \leqslant l \leqslant P$ and $1 \leqslant \mu \leqslant n_l$. These identities give equations (4.5) on the structure constants $\boldsymbol{\phi}_{[l;\lambda][l;\mu]}^{[l;\nu]}$ in (4.3). This completes the proof. □

4.1.2. Remark. The Verlinde-like formula (4.4) reproduces the classical Verlinde formula [50] for semisimple (rational) theories, in which the S-transformation diagonalizes the structure constants in a fusion algebra. Indeed, equations (4.5) take the following form

$$S^{\text{vac}}_{[l;1]}\boldsymbol{\phi}^{[l;1]}_{[l;1][l;1]} = 1,$$

and we get the Verlinde formula

$$N^{k}_{rs} \equiv N^{[k;1]}_{[r;1][s;1]} = \sum_{l=1}^{P} \frac{S_{rl}S_{sl}S_{lk}}{S^{\text{vac}}_{l}}.$$

4.2. *A generalized Verlinde formula for the* $(1, p)$ *models*

Here, we apply the results in 4.1.1 and 3.4 to obtain the structure constants ('generalized' fusion coefficients) with respect to the basis of the characters and pseudocharacters in $\mathsf{Z}_{\mathsf{cft}}$ from the modular-group action on $\mathsf{Z}_{\mathsf{cft}}$. These structure constants coincide with the structure constants in the Drinfeld basis (3.21),

$$\chi_{[r;\alpha]}\chi_{[s;\beta]} = \sum_{k=1}^{p+1}\sum_{\gamma=1}^{n_k} N^{[k;\gamma]}_{[r;\alpha][s;\beta]}\chi_{[k;\gamma]}, \tag{4.6}$$

where we write the Drinfeld basis (3.21) in the following 2-index notation parallel to the one in (2.5),

$$\begin{aligned}&\{\chi_{[s;\alpha]}|1 \leqslant s \leqslant p+1, 1 \leqslant \alpha \leqslant n_s\}\\ &\quad = \{\chi(1), \chi^{+}(1), \chi^{-}(p-1), \ldots, \chi(p-1), \chi^{+}(p-1), \chi^{-}(1), \chi^{+}(p), \chi^{-}(p)\},\end{aligned} \tag{4.7}$$

where $n_s = 3$, for $1 \leqslant s \leqslant p-1$, and $n_s = 1$, for $s = p, p+1$, that is we group the Drinfeld-basis elements into $(p-1)$ triplets $\{\chi(s), \chi^{+}(s), \chi^{-}(p-s)\}$, for $1 \leqslant s \leqslant p-1$, and into two singlets $\chi^{+}(p), \chi^{-}(p)$. Similarly, for the Radford basis (3.20), we denote

$$\begin{aligned}&\{\boldsymbol{\phi}_{[s;\alpha]}|1 \leqslant s \leqslant p+1, 1 \leqslant \alpha \leqslant n_s\}\\ &\quad = \{\boldsymbol{\phi}(1), \boldsymbol{\phi}^{+}(1), \boldsymbol{\phi}^{-}(p-1), \ldots, \boldsymbol{\phi}(p-1), \boldsymbol{\phi}^{+}(p-1), \boldsymbol{\phi}^{-}(1), \boldsymbol{\phi}^{+}(p), \boldsymbol{\phi}^{-}(p)\}.\end{aligned} \tag{4.8}$$

In [32], it was shown that the $SL(2, \mathbb{Z})$ action on the center Z is equivalent to the one on the space $\mathsf{Z}_{\mathsf{cft}}$ of torus amplitudes for the $(1, p)$ models. Therefore, the elements of the basis (4.7) can be linearly expressed with respect to the basis (4.8),

$$\chi_{[s;\alpha]} = \sum_{j=1}^{p+1}\sum_{\beta=1}^{n_j} S_{[s;\alpha][j;\beta]}\boldsymbol{\phi}_{[j;\beta]} \tag{4.9}$$

and vice versa

$$\boldsymbol{\phi}_{[s;\alpha]} = \sum_{j=1}^{p+1}\sum_{\beta=1}^{n_j} S_{[s;\alpha][j;\beta]}\chi_{[j;\beta]},$$

where the S-matrix elements $S_{[s;\alpha][j;\beta]}$ are given in (2.6)–(2.10).

Therefore, the structure constants $N^{[k;\gamma]}_{[r;\alpha][s;\beta]}$ in (4.6) can be calculated by using (4.4) and (4.5), where we must set $P = p+1, n_r = 3$, for $1 \leqslant r \leqslant p-1$, and $n_p = n_{p+1} = 1$; and the structure constants $\boldsymbol{\phi}_{[l;\lambda][l;\mu]}^{[l;\nu]}$ in the Radford basis (4.8) are solutions of (4.5) and coincide with (3.26)–(3.28). For $1 \leqslant l \leqslant p-1$, these structure constants are

$$\boldsymbol{\phi}_{[l;1][l;\mu]}^{[l;\nu]} = \frac{1}{S_{[l;1]}^{\mathrm{vac}}}\left(\sum_{i=1}^{3}\delta_{\mu,i}\delta_{\nu,i} - \frac{S_{[l;2]}^{\mathrm{vac}}}{S_{[l;1]}^{\mathrm{vac}}}\delta_{\mu,1}\delta_{\nu,2} - \frac{S_{[l;3]}^{\mathrm{vac}}}{S_{[l;1]}^{\mathrm{vac}}}\delta_{\mu,1}\delta_{\nu,3}\right), \tag{4.10}$$

and

$$\boldsymbol{\phi}_{[l;2][l;\mu]}^{[l;\nu]} = \frac{1}{S_{[l;1]}^{\mathrm{vac}}}\delta_{\mu,1}\delta_{\nu,2}, \qquad \boldsymbol{\phi}_{[l;3][l;\mu]}^{[l;\nu]} = \frac{1}{S_{[l;1]}^{\mathrm{vac}}}\delta_{\mu,1}\delta_{\nu,3}, \tag{4.11}$$

and, for $l = p, p+1$, $\boldsymbol{\phi}_{[l;1][l;\mu]}^{[l;\nu]} = \frac{1}{S_{[l;1]}^{\mathrm{vac}}}\delta_{\mu,1}\delta_{\nu,1}$. Hence, using (4.4), we finally get

$$N_{[r;\alpha][s;\beta]}^{[k;\gamma]} = \sum_{l=1}^{p+1}\sum_{\lambda=1}^{n_l}\frac{S_{[l;1]}^{\mathrm{vac}}S_{[l;1]}^{[r;\alpha]}S_{[l;\lambda]}^{[s;\beta]} + S_{[l;1]}^{\mathrm{vac}}S_{[l;\lambda]}^{[r;\alpha]}S_{[l;1]}^{[s;\beta]} - S_{[l;\lambda]}^{\mathrm{vac}}S_{[l;1]}^{[r;\alpha]}S_{[l;1]}^{[s;\beta]}}{\left(S_{[l;1]}^{\mathrm{vac}}\right)^2}S_{[k;\gamma]}^{[l;\lambda]}, \tag{4.12}$$

where we set $S_{[l;\lambda]}^{[r;\alpha]} \equiv S_{[r;\alpha][l;\lambda]}$. The generalized Verlinde formula (4.12) reproduces the structure constants in $\mathsf{Z}_{\mathrm{cft}}$ (given in 1.1.1) with respect to the basis (2.5) of the characters and pseudocharacters.

4.2.1. Remark. The structure constants (4.12), for $\alpha, \beta, \gamma = 2, 3$, coincide with the fusion coefficients of the $(1, p)$ models obtained in [38] and give the structure constants in the Grothendieck ring for $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ [32]. At the same time, the right-hand side of (4.12), for $\alpha, \beta, \gamma = 2, 3$, seems to be related to formula (5.28) from [30], giving the same fusion coefficients.

5. The space of boundary states and the center of the quantum group

Here, we analyze the boundary states in the $(1, p)$ models. In section 5.2, we choose a basis in the space of the boundary states in such a way that this basis corresponds to the Radford basis in the quantum group center Z. In section 5.3, we show that the states that correspond to the Drinfeld elements in Z satisfy the Cardy conditions and therefore are the Cardy boundary states.

5.1. Comments on Ishibashi, Cardy, Radford and Drinfeld boundary states

We call Ishibashi states any basis in the space of the boundary states $\mathsf{Z}_{\mathrm{cft}}$. Ishibashi states can be fixed by setting the matrix elements of the operator $q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}$ (see [37]) between them. We use the possibility in the following subsection to fix the Ishibashi states in such a way that they correspond to the Radford states in the center of the quantum group under an isomorphism $\mathcal{F} : \mathsf{Z} \to \mathsf{Z}_{\mathrm{cft}}$. Therefore, we call such chosen boundary states in $\mathsf{Z}_{\mathrm{cft}}$ the Radford boundary states.

Cardy states are the boundary states that satisfy the Cardy equation (see for example [37] formulas (2.13)–(2.15)). In this section, we show that under the isomorphism $\mathcal{F} : \mathsf{Z} \to \mathsf{Z}_{\mathrm{cft}}$ the Cardy states correspond to the Drinfeld states in Z.

5.2. Ishibashi and Radford boundary states

The equivalence between the triplet algebra $\mathcal{W}_p$- and $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$-representations categories leads to an isomorphism $\mathcal{F}$ between the center Z of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ and the space of the boundary states in the $(1, p)$ models. To describe the isomorphism $\mathcal{F}$, we first note that the space of the boundary states can be identified with the space $\mathsf{Z}_{\mathrm{cft}}$ of the vacuum torus amplitudes in the following way. We let $|\pm, s\rangle\!\rangle$ with $1 \leqslant s \leqslant p$ and $|s\rangle\!\rangle$ with $1 \leqslant s \leqslant p-1$ denote the Ishibashi states satisfying (cf [30])

$$\langle\!\langle r|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|s\rangle\!\rangle = \delta_{r,s}\frac{1}{S^{\mathrm{vac}}_{[s;1]}}\left(\chi_s(q) - \frac{S^{\mathrm{vac}}_{[s;2]}}{S^{\mathrm{vac}}_{[s;1]}}\chi^+_s(q) - \frac{S^{\mathrm{vac}}_{[s;3]}}{S^{\mathrm{vac}}_{[s;1]}}\chi^-_{p-s}(q)\right), \tag{5.1}$$

$$\langle\!\langle r|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|+, s\rangle\!\rangle = \delta_{r,s}\frac{1}{S^{\mathrm{vac}}_{[s;1]}}\chi^+_s(q), \tag{5.2}$$

$$\langle\!\langle r|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|-, p-s\rangle\!\rangle = \delta_{r,s}\frac{1}{S^{\mathrm{vac}}_{[s;1]}}\chi^-_{p-s}(q), \tag{5.3}$$

$$\langle\!\langle \alpha, r|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|\beta, s\rangle\!\rangle = 0, \qquad \alpha,\beta=\pm, \quad 1\leqslant r\leqslant p-1, \quad 1\leqslant s\leqslant p, \tag{5.4}$$

$$\langle\!\langle +, p|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|+, p\rangle\!\rangle = \frac{1}{S^{\mathrm{vac}}_{[p;1]}}\chi^+_p(q), \tag{5.5}$$

$$\langle\!\langle -, p|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|-, p\rangle\!\rangle = \frac{1}{S^{\mathrm{vac}}_{[p+1;1]}}\chi^-_p(q), \tag{5.6}$$

$$\langle\!\langle \pm, p|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|\mp, p\rangle\!\rangle = 0, \tag{5.7}$$

where the characters $\chi^\pm_s(q)$ are given in (A.1) and the pseudocharacters $\chi_s(q)$ in (A.2).

5.2.1. Remark. We note that the Ishibashi states proposed in [30] are in correspondence with those introduced in (5.1)–(5.7),

$$|P_t\rangle\!\rangle = S^{\mathrm{vac}}_{[t;1]}|t\rangle\!\rangle + S^{\mathrm{vac}}_{[t;2]}|+,t\rangle\!\rangle + S^{\mathrm{vac}}_{[t;3]}|-,p-t\rangle\!\rangle,$$
$$|U_t\rangle\!\rangle = |+,t\rangle\!\rangle + |-,p-t\rangle\!\rangle, \qquad |U^+_p\rangle\!\rangle = S^{\mathrm{vac}}_{[p;1]}|+,p\rangle\!\rangle, \qquad |U^-_p\rangle\!\rangle = S^{\mathrm{vac}}_{[p+1;1]}|-,p\rangle\!\rangle,$$

where we note that $S^{\mathrm{vac}}_{[t;2]} = S^{\mathrm{vac}}_{[t;3]}$. In what follows, we need all $(3p-1)$ Ishibashi states but not only $2p$ of them as in [30].

We next define the isomorphism $\mathcal{F}$ between the center Z and the space of boundary states, which we similarly denote as $\mathsf{Z}_{\mathrm{cft}}$,

$$\mathcal{F} : \mathsf{Z} \to \mathsf{Z}_{\mathrm{cft}}$$

by the formula

$$\mathcal{F}(\boldsymbol{\phi}^\pm(s)) = |\pm, s\rangle\!\rangle, \qquad \mathcal{F}(\boldsymbol{\phi}(s)) = |s\rangle\!\rangle. \tag{5.8}$$

We call the states $|\pm, s\rangle\!\rangle$ and $|s\rangle\!\rangle$ *the Radford boundary states*. Overlaps (5.1)–(5.7) between the Radford boundary states can be written shortly in the 2-index notation parallel to the one in (2.5),

$$\langle\!\langle [r;\alpha]|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|[s;\beta]\rangle\!\rangle = \sum_{l=1}^{p+1}\sum_{\gamma=1}^{n_l}\boldsymbol{\phi}^{[l;\gamma]}_{[r;\alpha][s;\beta]}\chi_{[l;\gamma]}(q) = \delta_{r,s}\sum_{\gamma=1}^{n_r}\boldsymbol{\phi}^{[r;\gamma]}_{[r;\alpha][r;\beta]}\chi_{[r;\gamma]}(q), \tag{5.9}$$

where $n_r = 3$, for $1 \leqslant r \leqslant p-1$, and $n_r = 1$, for $r = p, p+1$, and we set

$$\begin{aligned}&(|[s;1]\rangle\!\rangle, |[s;2]\rangle\!\rangle, |[s;3]\rangle\!\rangle) = (|s\rangle\!\rangle, |+,s\rangle\!\rangle, |-,p-s\rangle\!\rangle), \qquad 1\leqslant s\leqslant p-1,\\ &|[p;1]\rangle\!\rangle = |+,p\rangle\!\rangle, \qquad |[p+1;1]\rangle\!\rangle = |-,p\rangle\!\rangle,\end{aligned} \tag{5.10}$$

and analogously for the bra-vectors $\langle\!\langle [r;\alpha]|$; $\chi_{[r;\gamma]}(q)$ are defined in (2.5), and the structure constants $\boldsymbol{\phi}^{[r;\gamma]}_{[r;\alpha][r;\beta]}$ are given in (4.10) and (4.11).

5.3. Cardy and Drinfeld boundary states

We introduce the Drinfeld boundary states as the $\mathcal{F}$ images of the Drinfeld elements (3.21),

$$\mathcal{F}(\chi^{\pm}(s)) = \|\pm, s\rangle\!\rangle, \qquad \mathcal{F}(\chi(s)) = \|s\rangle\!\rangle.$$

We call the states $\|\pm, s\rangle\!\rangle$ and $\|s\rangle\!\rangle$ *the Drinfeld boundary states*. From (5.8) and (4.9), the Drinfeld boundary states are linearly expressed with respect to the Radford boundary states as

$$\|[s;\alpha]\rangle\!\rangle = \sum_{j=1}^{p+1}\sum_{\beta=1}^{n_j} S_{[s;\alpha][j;\beta]}|[j;\beta]\rangle\!\rangle, \tag{5.11}$$

where we also use the 2-index notation parallel to the one introduced in (5.10).

5.3.1. Proposition. *The Drinfeld boundary states (5.11) satisfy the Cardy condition*

$$\langle\!\langle[r;\alpha]\|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}\|[s;\beta]\rangle\!\rangle = \sum_{k=1}^{p+1}\sum_{\gamma=1}^{n_k} N_{[r;\alpha][s;\beta]}^{[k;\gamma]}\chi_{[k;\gamma]}(\widetilde{q}), \tag{5.12}$$

where $\widetilde{q} = \mathrm{e}^{-2\mathrm{i}\pi/\tau}$*, and the structure constants* $N_{[r;\alpha][s;\beta]}^{[k;\gamma]}$ *are given in (4.12).*

We note that the structure constants $N_{[r;\alpha][s;\beta]}^{[k;\gamma]}$ are integers and are non-negative whenever α, β and γ are not equal to 1.

Proof. This trivially follows from (5.11), (5.9) and (4.4). □

The $\mathcal{F}$ images of the Drinfeld elements in the quantum group center satisfy the Cardy condition (5.12) and are therefore the Cardy states.

5.3.2. Remark. We have $(3p-1)$ Drinfeld boundary states that satisfy the Cardy condition (5.12) but only $2p$ of them, $\|\pm, r\rangle\!\rangle$, for $1 \leqslant r \leqslant p$, have transparent physical meaning (see the last paragraph in section 5.1 [30]). Thus, we can identify the Drinfeld and Cardy boundary states. We also note that the Cardy states $\|\pm, r\rangle\!\rangle$ coincide with $\|(r,\pm)\rangle\!\rangle$ from [30].

6. Conclusions

In this paper, we propose a constructive method to study the boundary theories. The method is based on the well-known Kazhdan–Lusztig correspondence stated for the $(1, p)$ models in [32], and for (p, q)-logarithmic models in [33, 5]. The Kazhdan–Lusztig correspondence for the $(1, p)$ models is an equivalence [34] between the representation categories of the triplet algebra $\mathcal{W}_p$ and the *restricted* quantum group $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$, with $\mathfrak{q} = \mathrm{e}^{\frac{\mathrm{i}\pi}{p}}$. The equivalence leads to an isomorphism between the space of boundary states in the $(1, p)$ models and the center Z of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$.

We found a basis in the quantum-group center Z in which the structure constants are integer numbers and the $2p$ elements of the basis are the Drinfeld images of irreducible module characters, which span the Grothendieck ring. Under identification of the quantum group center and the space of $(1, p)$ model boundary states, the elements of the Drinfeld basis are mapped to the states satisfying the Cardy condition. Thus, we have $(3p-1)$ such states, but only $2p$ of them have transparent physical meaning and $(p-1)$ satisfy the Cardy condition only formally because the negative integer structure constants cannot be interpreted as multiplicities of any representations. These findings raise a good question about physical meaning of these additional $(p-1)$ Cardy states.

It is interesting to find the Radford and Drinfeld boundary states in terms of a free-scalar field, which is used for formulation of the $(1, p)$ models as screening kernels. It would allow us to understand better the meaning of $(p-1)$ boundary states $\|[r;1]\rangle\!\rangle$.

We also propose the generalized Verlinde formula (4.12), which gives the integer structure constants in the whole $(3p-1)$-dimensional space of vacuum torus amplitudes for the $(1, p)$ models, in which the fusion algebra is a $2p$-dimensional subalgebra (cf [49]) This formula can therefore be considered as a generalization of $(1, p)$ model Verlinde formulas derived in [30, 38, 49].

We hope that our results can be extended into the logarithmic (p, q) models [5], for which the Kazhdan–Lusztig dual quantum group is proposed in [33]. We thus conjecture that there are $\frac{1}{2}(3p-1)(3q-1)$ Cardy states in these (p, q) models. It would also be interesting to obtain a generalized Verlinde formula for the logarithmic (p, q) models and compare the results with the ones in [52, 9].

The subtle point is the braiding structure in the representation category of $\mathcal{W}_p$. It is known that the representation categories of $\mathcal{W}_p$ and $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ are equivalent as the Abelian categories [31]. Moreover, the tensor products of the irreducible and projective $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ modules coincide with the fusion of the corresponding $\mathcal{W}_p$ modules. This means that the representation categories should be equivalent as the tensor categories. In [35] it is shown that the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ representation category is not braided but the tensor products are commutative for *liftable* (in terminology of [35]) modules. Taking into account that fusion is always commutative by construction we can conclude that only $\mathcal{W}_p$ modules that correspond to liftable $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ modules can be realized in conformal field theory. Our opinion is that the final answer to this question can be done only in terms of a field theoretic construction for $\mathcal{W}_p$ modules.

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Appendix A. *W* characters and pseudocharacters

The $\mathcal{W}_p$ characters (1.2) are given by [38, 40]

$$\begin{aligned} \chi_s^+(q) &= \frac{1}{\eta(q)}\left(\frac{s}{p}\theta_{p-s,p}(q) + 2\theta'_{p-s,p}(q)\right), \\ \chi_s^-(q) &= \frac{1}{\eta(q)}\left(\frac{s}{p}\theta_{s,p}(q) - 2\theta'_{s,p}(q)\right), \end{aligned} \qquad 1 \leqslant s \leqslant p, \tag{A.1}$$

and the $\mathcal{W}_p$ pseudocharacters (2.3) are

$$\chi_s(q) = \frac{2a_0}{\eta(q)}\log(q)\theta'_{p-s,p}(q), \qquad 1 \leqslant s \leqslant p-1 \tag{A.2}$$

(see also [39]). Here, we use the eta function

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

and the theta functions

$$\theta_{s,p}(q,z) = \sum_{j\in\mathbb{Z}+\frac{s}{2p}} q^{pj^2} z^j, \qquad |q|<1, \qquad z\in\mathbb{C},$$

and set $\theta_{s,p}(q):=\theta_{s,p}(q,1)$ and $\theta'_{s,p}(q):=z\frac{\partial}{\partial z}\theta_{s,p}(q,z)|_{z=1}$.

Appendix B. Irreducible and projective $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ modules

Here, we recall the definition of the irreducible and projective $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ modules [32].

B.1. Irreducible $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ modules

The irreducible $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ modules are labeled by their highest weights $\mathfrak{q}^{s-1}$, where $s\in\mathbb{Z}/2p\mathbb{Z}$. We also parameterize the same highest weights as $\alpha\mathfrak{q}^{s-1}$, where $\alpha{=}\pm$ and $1\leqslant s\leqslant p$. Then, for $1\leqslant s\leqslant p$, the irreducible module with the highest weight $\pm\mathfrak{q}^{s-1}$ is denoted by $\mathsf{X}^{\pm}_s$. The dimension-s module $\mathsf{X}^{\pm}_s$ is spanned by elements $\mathsf{a}^{\pm}_n$, $0\leqslant n\leqslant s{-}1$, where $\mathsf{a}^{\pm}_0$ is the highest weight vector and the the left action of the algebra is given by

$$K\mathsf{a}^{\pm}_n = \pm\mathfrak{q}^{s-1-2n}\mathsf{a}^{\pm}_n, \qquad E\mathsf{a}^{\pm}_n = \pm[n][s-n]\mathsf{a}^{\pm}_{n-1}, \qquad F\mathsf{a}^{\pm}_n = \mathsf{a}^{\pm}_{n+1},$$

where we set $\mathsf{a}^{\pm}_{-1} = \mathsf{a}^{\pm}_s = 0$.

B.2. Projective $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$-modules

The module $\mathsf{P}^{\pm}_s$, $1\leqslant s\leqslant p-1$, is the projective module whose irreducible quotient is given by $\mathsf{X}^{\pm}_s$. Their structure can be schematically depicted as (see the explanation of notation in [34])

$$\begin{tikzcd} & \overset{\mathsf{X}^{\pm}_s}{\bullet} \arrow[dl] \arrow[dr] & \\ \overset{\mathsf{X}^{\mp}_{p-s}}{\bullet} \arrow[dr] & & \overset{\mathsf{X}^{\mp}_{p-s}}{\bullet} \arrow[dl] \\ & \overset{\mathsf{X}^{\pm}_s}{\bullet} & \end{tikzcd} \tag{B.1}$$

B.2.1. P^{+}_s. Let s be an integer $1\leqslant s\leqslant p-1$. The projective module P^{+}_s has the basis

$$\{\mathsf{t}^{(+,s)}_n, \mathsf{b}^{(+,s)}_n\}_{0\leqslant n\leqslant s-1} \cup \{\mathsf{l}^{(+,s)}_k, \mathsf{r}^{(+,s)}_k\}_{0\leqslant k\leqslant p-s-1}, \tag{B.2}$$

where $\{\mathsf{t}_n^{(+,s)}\}_{0\leqslant n\leqslant s-1}$ is the basis corresponding to the top module in (B.1), $\{\mathsf{b}_n^{(+,s)}\}_{0\leqslant n\leqslant s-1}$ to the bottom , $\{\mathsf{l}_k^{(+,s)}\}_{0\leqslant k\leqslant p-s-1}$ to the left, and $\{\mathsf{r}_k^{(+,s)}\}_{0\leqslant k\leqslant p-s-1}$ to the right module, with the $\overline{\mathscr{U}}_{\mathfrak{q}}s\ell(2)$ action given by

$$K\mathsf{l}_k^{(+,s)} = -\mathfrak{q}^{p-s-1-2k}\mathsf{l}_k^{(+,s)}, \qquad K\mathsf{r}_k^{(+,s)} = -\mathfrak{q}^{p-s-1-2k}\mathsf{r}_k^{(+,s)}, \qquad 0\leqslant k\leqslant p-s-1,$$

$$K\mathsf{b}_n^{(+,s)} = \mathfrak{q}^{s-1-2n}\mathsf{b}_n^{(+,s)}, \qquad K\mathsf{t}_n^{(+,s)} = \mathfrak{q}^{s-1-2n}\mathsf{t}_n^{(+,s)}, \qquad 0\leqslant n\leqslant s-1,$$

$$E\mathsf{l}_k^{(+,s)} = -[k][p-s-k]\mathsf{l}_{k-1}^{(+,s)}, \qquad 0\leqslant k\leqslant p-s-1 \quad (\text{with}\quad \mathsf{l}_{-1}^{(+,s)}\equiv 0),$$

$$E\mathsf{r}_k^{(+,s)} = \begin{cases} -[k][p-s-k]\mathsf{r}_{k-1}^{(+,s)}, & 1\leqslant k\leqslant p-s-1,\\ \mathsf{b}_{s-1}^{(+,s)}, & k=0,\end{cases}$$

$$E\mathsf{b}_n^{(+,s)} = [n][s-n]\mathsf{b}_{n-1}^{(+,s)}, \qquad 0\leqslant n\leqslant s-1 \quad (\text{with}\quad \mathsf{b}_{-1}^{(+,s)}\equiv 0),$$

$$E\mathsf{t}_n^{(+,s)} = \begin{cases} [n][s-n]\mathsf{t}_{n-1}^{(+,s)} + \mathsf{b}_{n-1}^{(+,s)}, & 1\leqslant n\leqslant s-1,\\ \mathsf{l}_{p-s-1}^{(+,s)}, & n=0,\end{cases}$$

and

$$F\mathsf{l}_k^{(+,s)} = \begin{cases} \mathsf{l}_{k+1}^{(+,s)}, & 0\leqslant k\leqslant p-s-2,\\ \mathsf{b}_0^{(+,s)}, & k=p-s-1,\end{cases}$$

$$F\mathsf{r}_k^{(+,s)} = \mathsf{r}_{k+1}^{(+,s)}, \qquad 0\leqslant k\leqslant p-s-1 \quad (\text{with}\quad \mathsf{r}_{p-s}^{(+,s)}\equiv 0),$$

$$F\mathsf{b}_n^{(+,s)} = \mathsf{b}_{n+1}^{(+,s)}, \qquad 0\leqslant n\leqslant s-1 \quad (\text{with}\quad \mathsf{b}_s^{(+,s)}\equiv 0),$$

$$F\mathsf{t}_n^{(+,s)} = \begin{cases} \mathsf{t}_{n+1}^{(+,s)}, & 0\leqslant n\leqslant s-2,\\ \mathsf{r}_0^{(+,s)}, & n=s-1.\end{cases}$$

B.2.2. P_{p-s}^{-}. Let s be an integer $1\leqslant s\leqslant p-1$. The projective module P_{p-s}^{-} has the basis

$$\{\mathsf{t}_k^{(-,s)}, \mathsf{b}_k^{(-,s)}\}_{0\leqslant k\leqslant p-s-1} \cup \{\mathsf{l}_n^{(-,s)}, \mathsf{r}_n^{(-,s)}\}_{0\leqslant n\leqslant s-1}, \tag{B.3}$$

where $\{\mathsf{t}_k^{(-,s)}\}_{0\leqslant k\leqslant p-s-1}$ is the basis corresponding to the top module in (B.1), $\{\mathsf{b}_k^{(-,s)}\}_{0\leqslant k\leqslant p-s-1}$ to the bottom, $\{\mathsf{l}_n^{(-,s)}\}_{0\leqslant n\leqslant s-1}$ to the left, and $\{\mathsf{r}_n^{(-,s)}\}_{0\leqslant n\leqslant s-1}$ to the right module, with the $\overline{\mathscr{U}}_{\mathfrak{q}}s\ell(2)$ action given by

$$K\mathsf{b}_k^{(-,s)} = -\mathfrak{q}^{p-s-1-2k}\mathsf{b}_k^{(-,s)}, \qquad K\mathsf{t}_k^{(-,s)} = -\mathfrak{q}^{p-s-1-2k}\mathsf{t}_k^{(-,s)}, \qquad 0\leqslant k\leqslant p-s-1,$$

$$K\mathsf{l}_n^{(-,s)} = \mathfrak{q}^{s-1-2n}\mathsf{l}_n^{(-,s)}, \qquad K\mathsf{r}_n^{(-,s)} = \mathfrak{q}^{s-1-2n}\mathsf{r}_n^{(-,s)}, \qquad 0\leqslant n\leqslant s-1,$$

$$E\mathsf{b}_k^{(-,s)} = -[k][p-s-k]\mathsf{b}_{k-1}^{(-,s)}, \qquad 0\leqslant k\leqslant p-s-1 \quad (\text{with}\quad \mathsf{b}_{-1}^{(-,s)}\equiv 0),$$

$$E\mathsf{t}_k^{(-,s)} = \begin{cases} -[k][p-s-k]\mathsf{t}_{k-1}^{(-,s)} + \mathsf{b}_{k-1}^{(-,s)}, & 1\leqslant k\leqslant p-s-1,\\ \mathsf{l}_{s-1}^{(-,s)}, & k=0,\end{cases}$$

$$E\mathsf{l}_n^{(-,s)} = [n][s-n]\mathsf{l}_{n-1}^{(-,s)}, \qquad 0\leqslant n\leqslant s-1 \quad (\text{with}\quad \mathsf{l}_{-1}^{(-,s)}\equiv 0),$$

$$E\mathsf{r}_n^{(-,s)} = \begin{cases} [n][s-n]\mathsf{r}_{n-1}^{(-,s)}, & 1\leqslant n\leqslant s-1,\\ \mathsf{b}_{p-s-1}^{(-,s)}, & n=0,\end{cases}$$

and $F\mathsf{b}_k^{(-,s)} = \mathsf{b}_{k+1}^{(-,s)}, \qquad 0\leqslant k\leqslant p-s-1 \quad (\text{with}\quad \mathsf{b}_{p-s}^{(-,s)}\equiv 0),$

$$F\mathsf{t}_k^{(-,s)} = \begin{cases} \mathsf{t}_{k+1}^{(-,s)}, & 0 \leqslant k \leqslant p-s-2, \\ \mathsf{r}_0^{(-,s)}, & k = p-s-1, \end{cases}$$

$$F\mathsf{l}_n^{(-,s)} = \begin{cases} \mathsf{l}_{n+1}^{(-,s)}, & 0 \leqslant n \leqslant s-2, \\ \mathsf{b}_0^{(-,s)}, & n = s-1, \end{cases}$$

$$F\mathsf{r}_n^{(-,s)} = \mathsf{r}_{n+1}^{(-,s)}, \qquad 0 \leqslant n \leqslant s-1 \quad (\text{with} \quad \mathsf{r}_s^{(-,s)} \equiv 0).$$

Appendix C. Radford images of pseudotraces and the proof of theorem 3.4

Here, we present bulky calculation of the Radford images of pseudotraces and give the proof of theorem 3.4 in appendix C.5.

C.1. PBW-basis action on projectives

The elements of the PBW basis of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ are enumerated as $E^i K^j F^\ell$ with $0 \leqslant i \leqslant p-1$, $0 \leqslant j \leqslant 2p-1$, $0 \leqslant \ell \leqslant p-1$. We calculate the PBW-basis action on the basis elements in the projective modules $\mathsf{P}_s^\pm$ (see (B.2) and (B.3)). Here, we closely follow the similar but more complicated calculation in [33]. We use the well-known identity (see, e.g., [53])

$$F^m E^m = \prod_{i=0}^{m-1} \left(C - \frac{\mathfrak{q}^{2i+1} K + \mathfrak{q}^{-2i-1} K^{-1}}{(\mathfrak{q} - \mathfrak{q}^{-1})^2} \right), \qquad m < p, \tag{C.1}$$

where the Casimir element is

$$C = FE + \frac{\mathfrak{q} K + \mathfrak{q}^{-1} K^{-1}}{(\mathfrak{q} - \mathfrak{q}^{-1})^2}.$$

Using (C.1), we calculate the action of $F^m E^m$ for $1 \leqslant m \leqslant p-1$ on $\mathsf{t}_n^{(+,s)}$ and $\mathsf{t}_k^{(-,s)}$ with the result

$$\begin{aligned} F^m E^m \mathsf{t}_n^{(+,s)} &= \prod_{i=0}^{m-1} \left(\mathsf{b}_n^{(+,s)} \frac{\partial}{\partial \mathsf{t}_n^{(+,s)}} + [s + \mathrm{i} - n][n - \mathrm{i}] \right) \mathsf{t}_n^{(+,s)} \\ &= \mathsf{B}_{n,m}^+(s) \mathsf{b}_n^{(+,s)} + \mathsf{T}_{n,m}^+(s) \mathsf{t}_n^{(+,s)}, \qquad 0 \leqslant n \leqslant s-1, \end{aligned} \tag{C.2}$$

and

$$\begin{aligned} F^m E^m \mathsf{t}_k^{(-,s)} &= \prod_{i=0}^{m-1} \left(\mathsf{b}_k^{(-,s)} \frac{\partial}{\partial \mathsf{t}_k^{(-,s)}} - [p - s + \mathrm{i} - k][k - \mathrm{i}] \right) \mathsf{t}_k^{(-,s)} \\ &= \mathsf{B}_{k,m}^-(p-s) \mathsf{b}_k^{(-,s)} + \mathsf{T}_{k,m}^-(p-s) \mathsf{t}_k^{(-,s)}, \qquad 0 \leqslant k \leqslant p-s-1, \end{aligned} \tag{C.3}$$

where the coefficients $\mathsf{B}_{n,m}^\pm(s)$ and $\mathsf{T}_{n,m}^\pm(s)$ are

- for $1 \leqslant m \leqslant n$,

$$\begin{aligned} \mathsf{B}_{n,m}^\pm(s) &= (\pm 1)^{m-1} \prod_{\mathrm{j}=n-m+1}^{n} [\mathrm{j}][s - \mathrm{j}] \sum_{\mathrm{i}=n-m+1}^{n} \frac{1}{[\mathrm{i}][s - \mathrm{i}]}, \\ \mathsf{T}_{n,m}^\pm(s) &= (\pm 1)^m \prod_{\mathrm{i}=n-m+1}^{n} [\mathrm{i}][s - \mathrm{i}], \end{aligned} \tag{C.4}$$

- for $n+1 \leqslant m \leqslant p-1$, $\mathsf{T}^+_{n,m}(s) = 0$,

$$\mathsf{B}^+_{n,m}(s) = (-1)^{m-n-1} \begin{bmatrix} s-1 \\ n \end{bmatrix} \begin{bmatrix} s-1+m-n \\ s \end{bmatrix} ([n]![m-n-1]!)^2, \tag{C.5}$$

and, for $k+1 \leqslant m \leqslant p-1$, $\mathsf{T}^-_{k,m}(p-s) = 0$,

$$\mathsf{B}^-_{k,m}(p-s) = (-1)^k \begin{bmatrix} s+k \\ k \end{bmatrix} \begin{bmatrix} s-1 \\ m-k-1 \end{bmatrix} ([k]![m-k-1]!)^2. \tag{C.6}$$

C.2. Idempotents and nilpotents

Here, we describe the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ center in terms of primitive idempotents and nilpotents. In theorem 3.4, we use this basis to calculate the structure constants in the center Z with respect to the Drinfeld and Radford bases.

C.2.1. Proposition [32]. *The center* Z *of* $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ *at* $\mathfrak{q} = e^{\frac{i\pi}{p}}$ *is* $(3p-1)$ *dimensional. Its associative commutative algebra structure is described as follows: there are two 'special' primitive idempotents* $\boldsymbol{e}_0$ *and* $\boldsymbol{e}_p$*,* $p-1$ *other primitive idempotents* $\boldsymbol{e}_s$, $1 \leqslant s \leqslant p-1$*, and* $2(p-1)$ *elements* $\boldsymbol{w}^\pm_s$ $(1 \leqslant s \leqslant p-1)$ *in the radical such that*

$$\boldsymbol{e}_s \boldsymbol{e}_{s'} = \delta_{s,s'} \boldsymbol{e}_s, \qquad s, s' = 0, \ldots, p, \tag{C.7}$$

$$\boldsymbol{e}_s \boldsymbol{w}^\pm_{s'} = \delta_{s,s'} \boldsymbol{w}^\pm_{s'}, \qquad 0 \leqslant s \leqslant p, \quad 1 \leqslant s' \leqslant p-1, \tag{C.8}$$

$$\boldsymbol{w}^\pm_s \boldsymbol{w}^\pm_{s'} = \boldsymbol{w}^\pm_s \boldsymbol{w}^\mp_{s'} = 0, \qquad 1 \leqslant s, \quad s' \leqslant p-1. \tag{C.9}$$

We fix the normalization of the nilpotents $\boldsymbol{w}^+_s$ and $\boldsymbol{w}^-_s$ such that they act as

$$\boldsymbol{w}^+_s \mathsf{t}^{(+,s)}_n = \mathsf{b}^{(+,s)}_n, \qquad \boldsymbol{w}^-_s \mathsf{t}^{(-,s)}_k = \mathsf{b}^{(-,s)}_k$$

in terms of the respective bases in the projective modules P^+_s and P^-_{p-s} defined in appendix B.1 and appendix B.2.

We call $\boldsymbol{e}_s$, $\boldsymbol{w}^\pm_s$ the *canonical central elements*.

C.2.2. Central elements decomposition. For any central element $A \in \mathsf{Z}$ and its decomposition

$$A = \sum_{s=0}^{p} a_s \boldsymbol{e}_s + \sum_{s=1}^{p-1} \left(c^+_s \boldsymbol{w}^+_s + c^-_s \boldsymbol{w}^-_s \right) \tag{C.10}$$

with respect to the canonical central elements, *the coefficient* a_s *is the eigenvalue of* A *in the irreducible representation* X^+_s*, the coefficient* c^+_s *is read off from the relation* $A\mathsf{t}^{(+,s)}_n = c^+_s \mathsf{b}^{(+,s)}_n$ *in* P^+_s*, and* c^-_s*, similarly, from the relation* $A\mathsf{t}^{(-,s)}_k = c^-_s \mathsf{b}^{(-,s)}_k$ *in* P^-_{p-s}.

C.3. Calculation of $\boldsymbol{\phi}(\gamma(s))$

The Radford map $\boldsymbol{\phi}$ on pseudotraces $\gamma(s)$ is given by

$$\boldsymbol{\phi}(\gamma(s)) = \boldsymbol{\phi}(s) = \sum_{(c)} \mathrm{Tr}_{\mathsf{P}^+_s \oplus \mathsf{P}^-_{p-s}} (K^{p-1} \boldsymbol{c}' \sigma_s) \boldsymbol{c}'',$$

where the map σ_s is defined in (3.5) and (3.10), the projective modules $\mathsf{P}^{\pm}_s$ are defined in appendix B.2 and

$$\Delta(\boldsymbol{c}) = \zeta \sum_{r=0}^{p-1}\sum_{s'=0}^{p-1}\sum_{j=0}^{2p-1}(-1)^{r+s'}\mathsf{q}^{-2(r+1)(s'+1)-r(r+1)-s'(s'+1)}$$
$$\times F^r E^{p-1-s'}K^{r-p+1+j}\otimes F^{p-1-r}E^{s'}K^{p-1-s'+j},$$

with $\zeta = \sqrt{\frac{p}{2}}\frac{1}{([p-1]!)^2}$. Using (C.2) and (C.3), we obtain

$$\boldsymbol{\phi}(s) = \boldsymbol{\Omega}^{+}(s) + \boldsymbol{\Omega}^{-}(p-s), \tag{C.11}$$

where we introduce

$$\boldsymbol{\Omega}^{+}(s) = \sum_{(\boldsymbol{c})}\mathrm{Tr}_{\mathsf{P}^{+}_{s}}(K^{p-1}\boldsymbol{c}'\sigma_s)\boldsymbol{c}''$$
$$= \alpha_s\zeta\sum_{m=0}^{p-2}\sum_{j=0}^{2p-1}\sum_{n=0}^{s-1}\mathsf{q}^{j(s-1-2n)}\mathsf{B}^{+}_{n,p-1-m}(s)F^mE^mK^j, \tag{C.12}$$

$$\boldsymbol{\Omega}^{-}(p-s) = \sum_{(\boldsymbol{c})}\mathrm{Tr}_{\mathsf{P}^{-}_{p-s}}(K^{p-1}\boldsymbol{c}'\sigma_s)\boldsymbol{c}''$$
$$= \alpha_s\zeta\sum_{m=0}^{p-2}\sum_{j=0}^{2p-1}\sum_{k=0}^{p-s-1}\mathsf{q}^{j(-s-1-2k)}\mathsf{B}^{-}_{k,p-1-m}(p-s)F^mE^mK^j, \tag{C.13}$$

with α_s given in (3.10), and the coefficients $\mathsf{B}^{+}_{n,m}(s)$ and $\mathsf{B}^{-}_{k,m}(p-s)$ are given in (C.4)–(C.6).

C.4. Proposition. *The Radford images on the pseudotraces $\gamma(s)$ are decomposed with respect to the canonical central elements (see C.2.1) as*

$$\boldsymbol{\phi}(s) = a_0(-1)^{p-s}\frac{\sqrt{2p}}{\mathsf{q}^s-\mathsf{q}^{-s}}\left(\boldsymbol{e}_s - \frac{\mathsf{q}^s+\mathsf{q}^{-s}}{[s]^2}\boldsymbol{w}_s\right),$$

where $\boldsymbol{w}_s = \boldsymbol{w}^{+}_s + \boldsymbol{w}^{-}_s$.

Proof. In the calculation, we closely follow the strategy proposed in appendix C.1. We use (C.11) to evaluate the action of $\boldsymbol{\phi}(s)$ on the modules $\mathsf{P}^{\pm}_{s'}$, $1 \leqslant s' \leqslant p-1$. This action is nonzero only on the module $\mathsf{P}^{+}_{s}\oplus\mathsf{P}^{-}_{p-s}$. Because $\boldsymbol{\phi}(s)$ is central, it suffices to evaluate the action in each direct summand only on any single vector, which we choose as $\mathsf{t}^{(\pm,s)}_0$. We first evaluate the action of $\boldsymbol{\Omega}^{+}(s)$ (see (C.11) and (C.12)) on $\mathsf{t}^{(+,s)}_0$ as

$$\boldsymbol{\Omega}^{+}(s)\mathsf{t}^{(+,s)}_0 = \alpha_s\zeta\sum_{n=0}^{s-1}\sum_{j=0}^{2p-1}\mathsf{q}^{j(s-1-2n)}\mathsf{B}^{+}_{n,p-1}(s)K^j\mathsf{t}^{(+,s)}_0 + \alpha_s\zeta\sum_{n=0}^{s-1}\sum_{m=1}^{p-2}\sum_{j=0}^{2p-1}\mathsf{q}^{j(s-1-2n)}$$
$$\times\mathsf{B}^{+}_{n,p-1-m}(s)K^j\prod_{r=1}^{m-1}\left(\boldsymbol{C} - \frac{\mathsf{q}^{2r+1}K+\mathsf{q}^{-2r-1}K^{-1}}{(\mathsf{q}-\mathsf{q}^{-1})^2}\right)\mathsf{b}^{(+,s)}_0,$$

where we use (C.1) and the formula $FE\mathsf{t}^{(+,s)}_0 = \mathsf{b}^{(+,s)}_0$ (see appendix B.1). Then, using (C.4) and (C.5), we obtain

$$\boldsymbol{\Omega}^{+}(s)\mathsf{t}^{(+,s)}_0 = \alpha_s(-1)^{p-s-1}\frac{p\sqrt{2p}}{[s]^2}\mathsf{t}^{(+,s)}_0$$

$$+\ \alpha_s 2p(-1)^p\zeta\Bigg(\sum_{m=1}^{s-1}(-1)^m\prod_{j=1}^{m}[j][s-j]\prod_{r=1}^{p-2-m}[r][s+r]\sum_{i=1}^{m}\frac{1}{[i][s-i]}$$
$$+(-1)^s\frac{[s-1]!}{[s]}\sum_{m=s}^{p-2}[m]![m-s]!\prod_{r=1}^{p-2-m}[r][s+r]\Bigg)\mathsf{b}_0^{(+,s)},$$

where we set $\sum_{m=1}^{0}\equiv 0$ and $\prod_{r=1}^{0}\equiv 1$, and the simple calculation gives

$$\boldsymbol{\Omega}^+(s)\mathsf{t}_0^{(+,s)} = \alpha_s(-1)^{p-s-1}\frac{p\sqrt{2p}}{[s]^2}\mathsf{t}_0^{(+,s)} + \alpha_s 2p(-1)^p\zeta(\boldsymbol{f}_{s,p}+\boldsymbol{g}_{s,p})\mathsf{b}_0^{(+,s)}, \tag{C.14}$$

where we introduce the following notation:

$$\begin{aligned}\boldsymbol{f}_{s,p} &= (-1)^{s-1}([s-1]![p-s-1]!)^2\sum_{i=1}^{s-1}\frac{1}{[i][s-i]},\\ \boldsymbol{g}_{s,p} &= (-1)^s\frac{[s-1]![p-s-1]!}{[s]}\sum_{m=1}^{p-s-1}\frac{[m+s]![p-s-1-m]!}{[m][s+m]},\end{aligned} \tag{C.15}$$

and the straightforward calculation gives us

$$\boldsymbol{g}_{s,p} = (-1)^{p-1}\boldsymbol{f}_{p-s,p}.$$

Therefore, from (C.14), we finally obtain

$$\boldsymbol{\Omega}^+(s)\mathsf{t}_0^{(+,s)} = \alpha_s(-1)^{p-s-1}\frac{p\sqrt{2p}}{[s]^2}\mathsf{t}_0^{(+,s)} + \alpha_s 2p\zeta((-1)^p\boldsymbol{f}_{s,p}-\boldsymbol{f}_{p-s,p})\mathsf{b}_0^{(+,s)}. \tag{C.16}$$

We next evaluate the action of $\boldsymbol{\Omega}^-(p-s)$ (see (C.11) and (C.13)) on $\mathsf{t}_0^{(+,s)}$ as

$$\begin{aligned}\boldsymbol{\Omega}^-(p-s)\mathsf{t}_0^{(+,s)} &= \alpha_s\zeta\sum_{k=0}^{p-s-1}\sum_{j=0}^{2p-1}q^{-2j(k+1)}\mathsf{B}^-_{k,p-1}(p-s)\mathsf{t}_0^{(+,s)}\\ &+\ \alpha_s\zeta\sum_{k=0}^{p-s-1}\sum_{m=1}^{p-2}\sum_{j=0}^{2p-1}q^{-2j(k+1)}\mathsf{B}^-_{k,p-1-m}(p-s)\\ &\times\prod_{r=1}^{m-1}\left(C-\frac{q^{2r+1}K+q^{-2r-1}K^{-1}}{(q-q^{-1})^2}\right)\mathsf{b}_0^{(+,s)} = 0,\end{aligned}$$

where we use the simple identity $\sum_{j=0}^{2p-1}q^{-2j(k+1)} = 0$. Hence, using (C.11), (C.15), and (C.16), and the identity

$$\sum_{i=1}^{p-s-1}\frac{1}{[i][s+i]} - \sum_{i=1}^{s-1}\frac{1}{[i][s-i]} = \frac{\mathfrak{q}^s+\mathfrak{q}^{-s}}{[s]^2},$$

we finally obtain the action of $\boldsymbol{\phi}(s)$ on $\mathsf{t}_0^{(+,s)}$ as

$$\boldsymbol{\phi}(s)\mathsf{t}_0^{(\pm,s)} = (-1)^{p-s}\frac{a_0}{\mathfrak{q}-\mathfrak{q}^{-1}}\frac{\sqrt{2p}}{[s]}\mathsf{t}_0^{(\pm,s)} + a_0(-1)^{p-s-1}\frac{\mathfrak{q}^s+\mathfrak{q}^{-s}}{\mathfrak{q}-\mathfrak{q}^{-1}}\frac{\sqrt{2p}}{[s]^3}\mathsf{b}_0^{(\pm,s)},$$

where we also give the result of the analogous calculation of the action of $\boldsymbol{\phi}(s)$ on $\mathsf{t}_0^{(-,s)}$. This completes the proof. □

C.5. The proof of theorem 3.4

We recall the definition of the canonical central elements (primitive idempotents $\boldsymbol{e}_s$ and nilpotents $\boldsymbol{w}_s^{\pm}$) given in appendix C.2.

C.5.1. Proposition [32]. [3] *For* $1 \leqslant s \leqslant p-1$,

$$\boldsymbol{\phi}^+(s) = \omega_s \boldsymbol{w}_s^+, \qquad \boldsymbol{\phi}^-(p-s) = \omega_s \boldsymbol{w}_s^-, \qquad \omega_s = (-1)^{p-s-1}\frac{p\sqrt{2p}}{[s]^2}, \tag{C.17}$$

$$\boldsymbol{\phi}^+(p) = p\sqrt{2p}\,\boldsymbol{e}_p, \qquad \boldsymbol{\phi}^-(p) = (-1)^{p+1}p\sqrt{2p}\,\boldsymbol{e}_0. \tag{C.18}$$

In proposition C.4, we evaluate $\boldsymbol{\phi}(s)$ as[4]

$$\boldsymbol{\phi}(s) = a_0(-1)^{p-s}\frac{\sqrt{2p}}{\mathfrak{q}^s - \mathfrak{q}^{-s}}\left(\boldsymbol{e}_s - \frac{\mathfrak{q}^s + \mathfrak{q}^{-s}}{[s]^2}\boldsymbol{w}_s\right), \tag{C.19}$$

where $\boldsymbol{w}_s = \boldsymbol{w}_s^+ + \boldsymbol{w}_s^-$.

Using (C.17)–(C.19) and (C.7)–(C.9), we calculate the multiplications (3.26)–(3.28) straightforwardly.

The multiplication (3.23) has been proven in [32].

C.5.2. Proposition. *For* $1 \leqslant s \leqslant p-1$,

$$\boldsymbol{\chi}(s) = a_0\frac{1}{\sqrt{2p}}\sum_{j=1}^{p-1}(-1)^{p+s+j}(\mathfrak{q}^{sj} - \mathfrak{q}^{-sj})\left(\frac{p-j}{p}\boldsymbol{\phi}^+(j) - \frac{j}{p}\boldsymbol{\phi}^-(p-j)\right). \tag{C.20}$$

Proof. We calculate $\boldsymbol{\chi}(s)$ using the $\mathcal{S}$-transformation (3.22), the identity $\mathcal{S}^2|_{\mathfrak{Z}} = \mathrm{id}$, and (C.19). This gives $\boldsymbol{\chi}(s)$ as

$$\boldsymbol{\chi}(s) = \mathcal{S}(\boldsymbol{\phi}(s)), \quad 1 \leqslant s \leqslant p-1. \tag{C.21}$$

In formula (C.19), the primitive idempotents $\boldsymbol{e}_s$ and the nilpotents $\boldsymbol{w}_s$ can be linearly expressed with respect to the Drinfeld basis as

$$\boldsymbol{e}_s = \frac{1}{2p^2}\Bigg(\sum_{j=1}^{p-1}(-1)^{j-1}((p+1-j)(\mathfrak{q}^{s(j-1)} + \mathfrak{q}^{-s(j-1)}) - (p-1-j)(\mathfrak{q}^{s(j+1)} + \mathfrak{q}^{-s(j+1)}))$$
$$\times(\boldsymbol{\chi}^+(j) + (-1)^{p-s}\boldsymbol{\chi}^-(j)) - (\mathfrak{q}^s + \mathfrak{q}^{-s})((-1)^{p-s}\boldsymbol{\chi}^+(p) + \boldsymbol{\chi}^-(p))\Bigg),$$

and

$$\boldsymbol{w}_s = (-1)^{p-s-1}\frac{[s]^2}{2p^2}\left(\sum_{j=1}^{p-1}(-1)^{p+s+j}(\mathfrak{q}^{js} + \mathfrak{q}^{-js})\boldsymbol{\kappa}(j) + \boldsymbol{\chi}^+(p) + (-1)^{p-s}\boldsymbol{\chi}^-(p)\right),$$

where we introduce the notation

$$\boldsymbol{\kappa}(j) = \boldsymbol{\chi}^+(j) + \boldsymbol{\chi}^-(p-j), \qquad 1 \leqslant j \leqslant p-1. \tag{C.22}$$

[3] We note a misprint in ([32], lemma 4.5.1): $\widehat{\boldsymbol{\phi}}^-(s)$ should be replaced by $\omega_{p-s}\boldsymbol{w}_{p-s}^-$ and ω_s should be replaced by $(-1)^{p-s-1}\frac{p\sqrt{2p}}{[s]^2}$.

[4] We note that $\boldsymbol{\phi}(s) = a_0\widehat{\rho}(s)$ in the terminology of ([32], section 5).

We thus obtain

$$\boldsymbol{\phi}(s) = a_0 \frac{1}{\sqrt{2p}} \sum_{r=1}^{p-1} (-1)^{r+s+p} (\mathfrak{q}^{rs} - \mathfrak{q}^{-rs}) \left(\frac{p-r}{p} \boldsymbol{\chi}^+(r) - \frac{r}{p} \boldsymbol{\chi}^-(p-r) \right). \tag{C.23}$$

Hence, we get $\boldsymbol{\chi}(s)$ as in (C.20) using (C.21) and (3.22). □

Proposition C.5.2 obviously gives, for $1 \leqslant r, s \leqslant p-1$,

$$\boldsymbol{\chi}(r)\boldsymbol{\chi}(s) = 0, \qquad \boldsymbol{\chi}^{\pm}(p)\boldsymbol{\chi}(s) = 0.$$

We next calculate multiplications (3.24) and (3.25).

C.5.3. Lemma. *We have, for* $1 \leqslant s \leqslant p-1$,

$$\boldsymbol{\chi}(s) = \boldsymbol{\chi}^+(s)\boldsymbol{\chi}(1), \tag{C.24}$$

$$\boldsymbol{\chi}(s) = -\boldsymbol{\chi}^-(p-s)\boldsymbol{\chi}(1). \tag{C.25}$$

Therefore, we have

$$\boldsymbol{\kappa}(r)\boldsymbol{\chi}(s) = 0, \tag{C.26}$$

with $\boldsymbol{\kappa}(s)$ *given in (C.22).*

Proof. We have, for $1 \leqslant s \leqslant p$,

$$\boldsymbol{\chi}^+(s) = s\boldsymbol{e}_p + (-1)^{s-1} s\boldsymbol{e}_0 + (-1)^s \times \sum_{j=1}^{p-1} \left(-\frac{[sj]}{[j]} \boldsymbol{e}_j + \frac{(s+1)[(s-1)j] - (s-1)[(s+1)j]}{[j]^3} \boldsymbol{w}_j \right), \tag{C.27}$$

and, for $0 \leqslant s \leqslant p-1$,

$$\boldsymbol{\chi}^-(p-s) = (p-s)\boldsymbol{e}_p + (-1)^{s-1}(p-s)\boldsymbol{e}_0 + (-1)^s \times \sum_{j=1}^{p-1} \left(\frac{[sj]}{[j]} \boldsymbol{e}_j + \frac{(p-s-1)[(s-1)j] - (p-s+1)[(s+1)j]}{[j]^3} \boldsymbol{w}_j \right), \tag{C.28}$$

with $\boldsymbol{w}_j = \boldsymbol{w}_j^+ + \boldsymbol{w}_j^-$. Hence, we obtain (C.24) and (C.25) by use of (C.20), (C.27), and (C.28). □

To prove (3.24) and (3.25), we use (3.23) and lemma C.5.3. This completes the proof of theorem 3.4.

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